4. Option pricing models under the Black-Scholes framework

*Riskless hedging principle*

Writer of a call option – hedges his exposure by holding certain units of the underlying asset in order to create a riskless portfolio.

In an efficient market with no riskless arbitrage opportunity, a riskless portfolio must earn rate of return equals the riskless interest rate.
**Dynamic replication strategy**

How to replicate an option dynamically by a portfolio of the riskless asset in the form of money market account and the risky underlying asset?

The cost of constructing the replicating portfolio gives the fair price of an option.

**Risk neutrality argument**

The two tradeable securities, option and asset, are hedgeable with each other. Hedgeable securities should have the same market price of risk.
Black-Scholes' assumptions on the financial market

(i) Trading takes place continuously in time.
(ii) The riskless interest rate $r$ is known and constant over time.
(iii) The asset pays no dividend.
(iv) There are no transaction costs in buying or selling the asset or the option, and no taxes.
(v) The assets are perfectly divisible.
(vi) There are no penalties to short selling and the full use of proceeds is permitted.
(vii) There are no arbitrage opportunities.
The stochastic process of the asset price $S$ is assumed to follow the Geometric Brownian motion

$$\frac{dS}{S} = \rho \, dt + \sigma \, dZ.$$ 

Consider a portfolio which involves short selling of one unit of a European call option and long holding of $\Delta$ units of the underlying asset. The value of the portfolio $\Pi$ is given by

$$\Pi = -c + \Delta S,$$

where $c = c(S, t)$ denotes the call price.

Since both $c$ and $\Pi$ are random variables, we apply the Ito lemma to compute their stochastic differentials as follows:

$$dc = \frac{\partial c}{\partial t} \, dt + \frac{\partial c}{\partial S} \, dS + \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} \, dt.$$
\[ d\Pi = -dc + \Delta dS \]
\[ = \left( -\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} \right) dt + \left( \Delta - \frac{\partial c}{\partial S} \right) dS \]
\[ = \left[ -\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + \left( \Delta - \frac{\partial c}{\partial S} \right) \rho S \right] dt + \left( \Delta - \frac{\partial c}{\partial S} \right) \sigma S \, dZ. \]

Why the differential \( Sd\Delta \) does not enter into \( d\Pi \)? By virtue of the assumption of following a self-financing trading strategy, the contribution to \( d\Pi \) due to \( Sd\Delta \) is offset by the accompanying purchase/sale of units of options.
If we choose $\Delta = \frac{\partial c}{\partial S}$, then the portfolio becomes a riskless hedge instantaneously \(\text{since } \frac{\partial c}{\partial S} \text{ changes continuously with time}\). By virtue of “no arbitrage”, the hedged portfolio should earn the riskless interest rate.

By setting $d\Pi = r\Pi dt$

$$d\Pi = \left( -\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} \right) dt = r\Pi dt = r \left( -c + S \frac{\partial c}{\partial S} \right) dt.$$  

Black-Scholes equation: \[
\frac{\partial c}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + rS \frac{\partial c}{\partial S} - rc = 0.
\]

Terminal payoff: $c(S,T) = \max(S - X, 0)$.

The parameter $\rho$ (expected rate of return) does not appear in the governing equation and the auxiliary condition.
• 5 parameters in the option model: \( S, T, X, r \) and \( \sigma \); only \( \sigma \) is unobservable.

**Deficiencies in the model**

1. Geometric Brownian motion assumption? Actual asset price dynamics is much more complicated.

2. Continuous hedging at all times
   — trading usually involves transaction costs.

3. Interest rate should be stochastic instead of deterministic.
Dynamic replication strategy (Merton’s approach)

\[ Q_S(t) = \text{number of units of asset} \]

\[ Q_V(t) = \text{number of units of option} \]

\[ M_S(t) = \text{dollar value of } Q_S(t) \text{ units of asset} \]

\[ M_V(t) = \text{dollar value of } Q_V(t) \text{ units of option} \]

\[ M(t) = \text{value of riskless asset invested in money market account} \]

- Construction of a self-financing and dynamically hedged portfolio containing risky asset, option and riskless asset.
• Dynamic replication: Composition is allowed to change at all times in the replication process.

• The self-financing portfolio is set up with zero initial net investment cost and no additional funds added or withdrawn afterwards.

The zero net investment condition at time \( t \) is

\[
\Pi(t) = M_S(t) + M_V(t) + M(t) = Q_S(t)S + Q_V(t)V + M(t) = 0.
\]

Differential of option value \( V \):

\[
dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} dt
\]

\[
= \left( \frac{\partial V}{\partial t} + \rho S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dZ.
\]
Formally, we write

\[
\frac{dV}{V} = \rho_V \, dt + \sigma_V \, dZ
\]

where

\[
\rho_V = \frac{\frac{\partial V}{\partial t} + \rho_S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2}}{V} \quad \text{and} \quad \sigma_V = \frac{\sigma S \frac{\partial V}{\partial S}}{V}.
\]

\[
d\Pi(t) = \left[ Q_S(t) \, dS + Q_V(t) \, dV + rM(t) \, dt \right] + \left[ S \, dQ_S(t) + V \, dQ_S(t) + dM(t) \right]
\]

zero due to self-financing trading strategy
The instantaneous portfolio return $d\Pi(t)$ can be expressed in terms of $M_S(t)$ and $M_V(t)$ as follows:

\[
d\Pi(t) = Q_S(t) \frac{dS}{S} + Q_V(t) \frac{dV}{V} + rM(t) dt
\]

\[
= M_S(t) \frac{dS}{S} + M_V(t) \frac{dV}{V} + rM(t) dt
\]

\[
= [(\rho - r)M_S(t) + (\rho_V - r)M_V(t)] dt
\]

\[
+ [\sigma M_S(t) + \sigma_V M_V(t)] dZ.
\]

We then choose $M_S(t)$ and $M_V(t)$ such that the stochastic term becomes zero.

From the relation:

\[
\sigma M_S(t) + \sigma_V M_V(t) = \sigma S Q_S(t) + \frac{\sigma S \partial V}{V} Q_V(t) = 0,
\]

we obtain

\[
\frac{Q_S(t)}{Q_V(t)} = -\frac{\partial V}{\partial S}.
\]
Taking the choice of $Q_V(t) = -1$, and knowing

$$0 = \Pi(t) = -V + \Delta S + M(t)$$

we obtain

$$V = \Delta S + M(t), \text{ where } \Delta = \frac{\partial V}{\partial S}.$$ 

Since the replicating portfolio is self-financing and replicates the terminal payoff, by virtue of no-arbitrage argument, the initial cost of setting up this replicating portfolio of risky asset and riskless asset must be equal to the value of the option being replicated.
The dynamic replicating portfolio is riskless and requires no net investment, so $d\Pi(t) = 0$.

$$0 = [(\rho - r)M_S(t) + (\rho V - r)M_V(t)] dt.$$  

Putting $\frac{Q_S(t)}{Q_V(T)} = -\frac{\partial V}{\partial S}$, we obtain

$$(\rho - r)S\frac{\partial V}{\partial S} = (\rho V - r)V.$$  

Replacing $\rho V$ by $\left[\frac{\partial V}{\partial t} + \rho S\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}\right] / V$, we obtain the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$
Alternative perspective on risk neutral valuation

From $\rho V = \frac{\partial V}{\partial t} + \rho S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2}$, we obtain

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \rho S \frac{\partial V}{\partial S} - \rho V V = 0.$$ 

We need to calibrate the parameters $\rho$ and $\rho V$, or find some other means to avoid such nuisance.

Combining $\sigma V = \frac{\sigma S \partial V}{V}$ and $(\rho - r) S \frac{\partial V}{\partial S} = (\rho V - r) V$, we obtain

$$\frac{\rho V - r}{\sigma V} = \frac{\rho - r}{\lambda V} \quad \Rightarrow \quad \text{Black-Scholes equation.}$$
The market price of risk is the rate of extra return above \( r \) per unit risk.

Two hedgeable securities should have the same market price of risk.

The Black-Scholes equation can be obtained by setting \( \rho = \rho_V = r \) (implying zero market price of risk).

In the world of zero market price of risk, investors are said to be risk neutral since they do not demand extra returns on holding risky assets.

Option valuation can be performed in the risk neutral world by artificially taking the expected rate of returns of the asset and option to be \( r \).
Arguments of risk neutrality

- We find the price of a derivative *relative* to that of the underlying asset $\Rightarrow$ mathematical relationship between the prices is invariant to the risk preference.

- Be careful that the actual rate of return of the underlying asset would affect the asset price and thus indirectly affects the *absolute* derivative price.

- We simply use the convenience of risk neutrality to arrive at the mathematical relationship but actual risk neutrality behaviors of the investors are not necessary in the derivation of option prices.
“How we came up with the option formula?” — Black (1989)

- It started with tinkering and ended with delayed recognition.
- The expected return on a warrant should depend on the risk of the warrant in the same way that a common stock’s expected return depends on its risk.
- I spent many, many days trying to find the solution to that (differential) equation. I have a PhD in applied mathematics, but had never spent much time on differential equations, so I didn’t know the standard methods used to solve problems like that. I have an A.B. in physics, but I didn’t recognize the equation as a version of the heat equation, which has well-known solutions.
Continuous time securities model

- Uncertainty in the financial market is modeled by the filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)\), where \(\Omega\) is a sample space, \(\mathcal{F}\) is a \(\sigma\)-algebra on \(\Omega\), \(P\) is a probability measure on \((\Omega, \mathcal{F})\), \(\mathcal{F}_t\) is the filtration and \(\mathcal{F}_T = \mathcal{F}\).

- There are \(M + 1\) securities whose price processes are modeled by adapted stochastic processes \(S_m(t), m = 0, 1, \cdots, M\).

- We define \(h_m(t)\) to be the number of units of the \(m^{\text{th}}\) security held in the portfolio.

- The trading strategy \(H(t)\) is the vector stochastic process \((h_0(t) h_1(t) \cdots h_M(t))^T\), where \(H(t)\) is a \((M+1)\)-dimensional predictable process since the portfolio composition is determined by the investor based on the information available before time \(t\).
• The value process associated with a trading strategy $H(t)$ is defined by

$$V(t) = \sum_{m=0}^{M} h_m(t) S_m(t), \quad 0 \leq t \leq T,$$

and the gain process $G(t)$ is given by

$$G(t) = \sum_{m=0}^{M} \int_{0}^{t} h_m(u) \, dS_m(u), \quad 0 \leq t \leq T.$$ 

• Similar to that in discrete models, $H(t)$ is self-financing if and only if

$$V(t) = V(0) + G(t).$$
• We use $S_0(t)$ to denote the money market account process that grows at the riskless interest rate $r(t)$, that is,

$$dS_0(t) = r(t)S_0(t) ~ dt.$$ 

• The discounted security price process $S^*_m(t)$ is defined as

$$S^*_m(t) = S_m(t)/S_0(t), \quad m = 1, 2, \ldots, M.$$ 

• The discounted value process $V^*(t)$ is defined by dividing $V(t)$ by $S_0(t)$. The discounted gain process $G^*(t)$ is defined by

$$G^*(t) = V^*(t) - V^*(0).$$
No-arbitrage principle and equivalent martingale measure

- A self-financing trading strategy $H$ represents an arbitrage opportunity if and only if (i) $G^*(T) \geq 0$ and (ii) $E_P G^*(T) > 0$ where $P$ is the actual probability measure of the states of occurrence associated with the securities model.

- A probability measure $Q$ on the space $(\Omega, \mathcal{F})$ is said to be an equivalent martingale measure if it satisfies

(i) $Q$ is equivalent to $P$, that is, both $P$ and $Q$ have the same null set;
(ii) the discounted security price processes $S^*_m(t), m = 1, 2, \cdots, M$ are martingales under $Q$, that is,

$$E_Q[S^*_m(u)|\mathcal{F}_t] = S^*_m(t), \quad \text{for all } 0 \leq t \leq u \leq T.$$
Theorem

Let $Y$ be an attainable contingent claim generated by some trading strategy $H$ and assume that an equivalent martingale measure $Q$ exists, then for each time $t, 0 \leq t \leq T$, the arbitrage price of $Y$ is given by

$$V(t; H) = S_0(t)E_Q \left[ \frac{Y}{S_0(T)} \bigg| \mathcal{F}_t \right].$$

The validity of the Theorem is readily seen if we consider the discounted value process $V^*(t; H)$ to be a martingale under $Q$. This leads to

$$V(t; H) = S_0(t)V^*(t; H) = S_0(t)E_Q[V^*(T; H) | \mathcal{F}_t].$$

Furthermore, by observing that $V^*(T; H) = Y/S_0(T)$, so the risk neutral valuation formula follows.
Change of numeraire

- The choice of $S_0(t)$ as the numeraire is not unique in order that the risk neutral valuation formula holds.

- Let $N(t)$ be a numeraire whereby we have the existence of an equivalent probability measure $Q_N$ such that all security prices discounted with respect to $N(t)$ are $Q_N$-martingale. In addition, if a contingent claim $Y$ is attainable under $(S_0(t), Q)$, then it is also attainable under $(N(t), Q_N)$. 
The arbitrage price of any security given by the risk neutral valuation formula under both measures should agree. We then have

\[ S_0(t)E_Q \left[ \frac{Y}{S_0(T)} \mid \mathcal{F}_t \right] = N(t)E_{Q_N} \left[ \frac{Y}{N(T)} \mid \mathcal{F}_t \right]. \]

To effect the change of measure from \( Q_N \) to \( Q \), we multiply by the Radon-Nikodym derivative so that

\[ S_0(t)E_Q \left[ \frac{Y}{S_0(T)} \mid \mathcal{F}_t \right] = N(t)E_Q \left[ \frac{Y}{N(T)} \frac{dQ_N}{dQ} \mid \mathcal{F}_t \right]. \]

By comparing like terms, we obtain

\[ \frac{dQ_N}{dQ} = \frac{N(T)}{N(t)} \bigg/ \frac{S_0(T)}{S_0(t)}. \]
Black-Scholes model revisited

The price processes of $S(t)$ and $M(t)$ are governed by

\[
\begin{align*}
\frac{dS(t)}{S(t)} &= \rho \ dt + \sigma \ dZ \\
\text{and} \\
dM(t) &= rM(t) \ dt.
\end{align*}
\]

The price process of $S^*(t) = S(t)/M(t)$ becomes

\[
\frac{dS^*(t)}{S^*(t)} = (\rho - r)dt + \sigma \ dZ.
\]

We would like to find the equivalent martingale measure $Q$ such that the discounted asset price $S^*$ is $Q$-martingale. By the Girsanov Theorem, suppose we choose $\gamma(t)$ in the Radon-Nikodym derivative such that

\[
\gamma(t) = \frac{\rho - r}{\sigma},
\]

then $\tilde{Z}$ is a Brownian motion under the probability measure $Q$ and

\[
d\tilde{Z} = dZ + \frac{\rho - r}{\sigma}dt.
\]
Under the $Q$-measure, the process of $S^*(t)$ now becomes

$$\frac{dS^*(t)}{S^*(t)} = \sigma \ d\tilde{Z},$$

hence $S^*(t)$ is $Q$-martingale. The asset price $S(t)$ under the $Q$-measure is governed by

$$\frac{dS(t)}{S(t)} = r \ dt + \sigma \ d\tilde{Z}.$$ 

When the money market account is used as the numeraire, the corresponding equivalent martingale measure is called the risk neutral measure and the drift rate of $S$ under the $Q$-measure is called the risk neutral drift rate.
The arbitrage price of a derivative is given by

$$V(S, t) = e^{-r(T-t)}E^t_S[\hat{h}(S_T)]$$

where $E^t_S$ is the expectation under the risk neutral measure $Q$ conditional on the filtration $\mathcal{F}_t$ and $S_t = S$. By the Feynman-Kac representation formula, the governing equation of $V(S, t)$ is given by

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$ 

Consider the European call option whose terminal payoff is $\max(S_T - X, 0)$. The call price $c(S, t)$ is given by

$$c(S, t) = e^{-r(T-t)}E_Q[\max(S_T - X, 0)]$$

$$= e^{-r(T-t)}\{E_Q[S_T \mathbf{1}_{\{S_T \geq X\}}] - XE_Q[\mathbf{1}_{\{S_T \geq X\}}]\}.$$
Exchange rate process under domestic risk neutral measure

• Consider a foreign currency option whose payoff function depends on the exchange rate $F$, which is defined to be the domestic currency price of one unit of foreign currency.

• Let $M_d$ and $M_f$ denote the money market account process in the domestic market and foreign market, respectively. The processes of $M_d(t)$, $M_f(t)$ and $F(t)$ are governed by

$$dM_d(t) = r M_d(t) \, dt, \quad dM_f(t) = r_f M_f(t) \, dt, \quad \frac{dF(t)}{F(t)} = \mu \, dt + \sigma \, dZ_F,$$

where $r$ and $r_f$ denote the riskless domestic and foreign interest rates, respectively.
• We may treat the domestic money market account and the foreign money market account in domestic dollars (whose value is given by $FM_f$) as traded securities in the domestic currency world.

• With reference to the domestic equivalent martingale measure, $M_d$ is used as the numeraire.

• By Ito’s lemma, the relative price process $X(t) = F(t)M_f(t)/M_d(t)$ is governed by

$$\frac{dX(t)}{X(t)} = (r_f - r + \mu) dt + \sigma dZ_F.$$
• With the choice of \( \gamma = (r_f - r + \mu)/\sigma \) in the Girsanov Theorem, we define

\[
dZ_d = dZ_F + \gamma \, dt,
\]

where \( Z_d \) is a Brownian process under \( Q_d \).

• Under the domestic equivalent martingale measure \( Q_d \), the process of \( X \) now becomes

\[
\frac{dX(t)}{X(t)} = \sigma \, dZ_d
\]

so that \( X \) is \( Q_d \)-martingale.

• The exchange rate process \( F \) under the \( Q_d \)-measure is given by

\[
\frac{dF(t)}{F(t)} = (r - r_f) \, dt + \sigma \, dZ_d.
\]

• The risk neutral drift rate of \( F \) under \( Q_d \) is found to be \( r - r_f \).
Recall that the Black-Scholes equation for a European vanilla call option takes the form

\[
\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + rS \frac{\partial c}{\partial S} - rc, \quad 0 < S < \infty, \quad \tau > 0, \quad \tau = T - t.
\]

**Initial condition (payoff at expiry)**

\[c(S, 0) = \max(S - X, 0), \quad X \text{ is the strike price.}\]

Using the transformation: \(y = \ln S\) and \(c(y, \tau) = e^{-r\tau}w(y, \tau)\), the Black-Scholes equation is transformed into

\[
\frac{\partial w}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial y^2} + \left(r - \frac{\sigma^2}{2}\right) \frac{\partial w}{\partial y}, \quad -\infty < y < \infty, \quad \tau > 0.
\]

The initial condition for the model now becomes

\[w(y, 0) = \max(e^y - X, 0).\]
Green function approach

The infinite domain Green function is known to be

\[ \phi(y, \tau) = \frac{1}{\sigma \sqrt{2\pi \tau}} \exp \left( -\frac{[y + (r - \frac{\sigma^2}{2})\tau]^2}{2\sigma^2\tau} \right). \]

Here, \( \phi(y, \tau) \) satisfies the initial condition:

\[ \lim_{\tau \to 0^+} \phi(y, \tau) = \delta(y), \]

where \( \delta(y) \) is the Dirac function representing a unit impulse at the origin.

The initial condition can be expressed as

\[ w(y, 0) = \int_{-\infty}^{\infty} w(\xi, 0)\delta(y - \xi) \, d\xi, \]

so that \( w(y, 0) \) can be considered as the superposition of impulses with varying magnitude \( w(\xi, 0) \) ranging from \( \xi \to -\infty \) to \( \xi \to \infty \).
Since the Black-Scholes equation is linear, the response in position \( y \) and at time to expiry \( \tau \) due to an impulse of magnitude \( w(\xi, 0) \) in position \( \xi \) at \( \tau = 0 \) is given by \( w(\xi, 0)\phi(y - \xi, \tau) \).

From the principle of superposition for a linear differential equation, the solution is obtained by summing up the responses due to these impulses.

\[
c(y, \tau) = e^{-r\tau}w(y, \tau) = e^{-r\tau} \int_{-\infty}^{\infty} w(\xi, 0) \phi(y - \xi, \tau) \, d\xi
\]

\[
= e^{-r\tau} \int_{\ln X}^{\infty} (e^{\xi} - X) \frac{1}{\sigma \sqrt{2\pi\tau}} \exp \left( - \frac{[y + (r - \frac{\sigma^2}{2})\tau - \xi]^2}{2\sigma^2\tau} \right) \, d\xi.
\]
Note that
\[
\int_\ln X^\infty e^\xi \frac{1}{\sigma \sqrt{2\pi \tau}} \exp \left(- \frac{\left[y + (r - \frac{\sigma^2}{2})\tau - \xi\right]^2}{2\sigma^2\tau}\right) d\xi
\]
\[
= \exp(y + r\tau) \int_\ln X^\infty \frac{1}{\sigma \sqrt{2\pi \tau}} \exp \left(- \frac{\left[y + (r + \frac{\sigma^2}{2})\tau - \xi\right]^2}{2\sigma^2\tau}\right) d\xi
\]
\[
= e^{r\tau} SN \left(\frac{\ln \frac{S}{X} + (r + \frac{\sigma^2}{2})\tau}{\sigma \sqrt{\tau}}\right), \quad y = \ln S;
\]
\[
\int_\ln X^\infty \frac{1}{\sigma \sqrt{2\pi \tau}} \exp \left(- \frac{\left[y + (r - \frac{\sigma^2}{2})\tau - \xi\right]^2}{2\sigma^2\tau}\right) d\xi
\]
\[
= N \left(\frac{y + (r - \frac{\sigma^2}{2})\tau - \ln X}{\sigma \sqrt{\tau}}\right) = N \left(\frac{\ln \frac{S}{X} + (r - \frac{\sigma^2}{2})\tau}{\sigma \sqrt{\tau}}\right), \quad y = \ln S.
\]
Hence, the price formula of the European call option is found to be

\[ c(S, \tau) = SN(d_1) - X e^{-r\tau} N(d_2), \]

where

\[ d_1 = \frac{\ln \frac{S}{X} + (r + \frac{\sigma^2}{2})\tau}{\sigma \sqrt{\tau}}, \quad d_2 = d_1 - \sigma \sqrt{\tau}. \]

The call value lies within the bounds

\[ \max(S - X e^{-r\tau}, 0) \leq c(S, \tau) \leq S, \quad S \geq 0, \tau \geq 0, \]
\[ c(S, \tau) = e^{-r\tau} E_Q[(S_T - X) \mathbf{1}_{\{S_T \geq X\}}] \]
\[ = e^{-r\tau} \int_0^\infty \max(S_T - X, 0) \psi(S_T, T; S, t) \ dS_T. \]

- Under the risk neutral measure,

\[ \ln \frac{S_T}{S} = \left( r - \frac{\sigma^2}{2} \right) \tau + \sigma \tilde{Z}(\tau) \]

so that \( \ln \frac{S_T}{S} \) is normally distributed with mean \( \left( r - \frac{\sigma^2}{2} \right) \tau \) and variance \( \sigma^2 \tau, \tau = T - t. \)

- From the density function of a normal random variable, the transition density function is given by

\[ \psi(S_T, T; S, t) = \frac{1}{S_T \sigma \sqrt{2\pi \tau}} \exp \left( - \frac{\left( \ln \frac{S_T}{S} - \left( r - \frac{\sigma^2}{2} \right) \tau \right)^2}{2\sigma^2 \tau} \right). \]
If we compare the price formula with the expectation representation we deduce that

\[ N(d_2) = E_Q[1_{\{S_T \geq X\}}] = Q[S_T \geq X] \]
\[ SN(d_1) = e^{-r\tau} E_Q[S_T 1_{\{S_T \geq X\}}]. \]

• \(N(d_2)\) is recognized as the probability under the risk neutral measure \(Q\) that the call expires in-the-money, so \(Xe^{-r\tau} N(d_2)\) represents the present value of the risk neutral expectation of payment paid by the option holder at expiry.

• \(SN(d_1)\) is the discounted risk neutral expectation of the terminal asset price conditional on the call being in-the-money at expiry.
Delta - derivative with respect to asset price

\[
\Delta_c = \frac{\partial c}{\partial S} = N(d_1) + S \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \frac{\partial d_1}{\partial S} - X e^{-r\tau} \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \frac{\partial d_2}{\partial S}
\]

\[
= N(d_1) + \frac{1}{\sigma \sqrt{2\pi \tau}} \left[ e^{-\frac{d_1^2}{2}} - e^{-(r\tau + \ln S) - \frac{d_2^2}{2}} \right]
\]

\[
= N(d_1) > 0.
\]

Knowing that a European call can be replicated by \( \Delta \) units of asset and riskless asset in the form of money market account, the factor \( N(d_1) \) in front of \( S \) in the call price formula thus gives the hedge ratio \( \Delta \).
\( \Delta_c \) is an increasing function of \( S \) since \( \frac{\partial}{\partial S} N(d_1) \) is always positive. Also, the value of \( \Delta_c \) is bounded between 0 and 1.

- The curve of \( \Delta_c \) against \( S \) changes concavity at

\[
S_c = X \exp \left( - \left( r + \frac{3\sigma^2}{2} \right) \tau \right)
\]

so that the curve is concave upward for \( 0 \leq S < S_c \) and concave downward for \( S_c < S < \infty \).

\[
\lim_{\tau \to \infty} \frac{\partial c}{\partial S} = 1 \quad \text{for all values of } S,
\]

while

\[
\lim_{\tau \to 0^+} \frac{\partial c}{\partial S} = \begin{cases} 
1 & \text{if } S > X \\
\frac{1}{2} & \text{if } S = X \\
0 & \text{if } S < X 
\end{cases}
\]
Variation of the delta of the European call value with respect to the asset price $S$. The curve changes concavity at $S = X e^{-\left(r + \frac{3\sigma^2}{2}\right)\tau}$. 
Variation of the delta of the European call value with respect to time to expiry $\tau$. The delta value always tends to one from below when the time to expiry tends to infinity. The delta value tends to different asymptotic limits as time comes close to expiry, depending on the moneyness of the option.
Continuous dividend yield models

Let $q$ denote the constant continuous dividend yield, that is, the holder receives dividend of amount equal to $qS \, dt$ within the interval $dt$. The asset price dynamics is assumed to follow the Geometric Brownian Motion

$$\frac{dS}{S} = \rho \, dt + \sigma \, dZ.$$ 

We form a riskless hedging portfolio by short selling one unit of the European call and long holding $\triangle$ units of the underlying asset. The differential of the portfolio value $\Pi$ is given by
\[ d\Pi = -dc + \partial dS + q\partial S \, dt \]
\[ = \left( -\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + q\partial S \right) dt + \left( \partial - \frac{\partial c}{\partial S} \right) dS. \]

The last term \( q\partial S \, dt \) is the wealth added to the portfolio due to the dividend payment received. By choosing \( \partial = \frac{\partial c}{\partial S} \), we obtain a riskless hedge for the portfolio. The hedged portfolio should earn the riskless interest rate.

We then have
\[ d\Pi = \left( -\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + qS\frac{\partial c}{\partial S} \right) dt = r \left( -c + S\frac{\partial c}{\partial S} \right) dt, \]
which leads to
\[
\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + (r - q)S\frac{\partial c}{\partial S} - rc, \quad \tau = T - t, \quad 0 < S < \infty, \quad \tau > 0.
\]
Martingale pricing approach

Suppose all the dividend yields received are used to purchase additional units of asset, then the wealth process of holding one unit of asset initially is given by

$$\hat{S}_t = e^{qt} S_t,$$

where $e^{qt}$ represents the growth factor in the number of units. The wealth process $\hat{S}_t$ follows

$$\frac{d\hat{S}_t}{\hat{S}_t} = (\rho + q) \, dt + \sigma \, dZ.$$

We would like to find the equivalent risk neutral measure $Q$ under which the discounted wealth process $\hat{S}_t^*$ is $Q$-martingale. We choose $\gamma(t)$ in the Radon-Nikodym derivative to be

$$\gamma(t) = \frac{\rho + q - r}{\sigma}.$$
Now \( \tilde{Z} \) is Brownian process under \( Q \) and
\[
d\tilde{Z} = dZ + \frac{\rho + q - r}{\sigma} dt.
\]
Also, \( \hat{S}_t^* \) becomes \( Q \)-martingale since
\[
\frac{d\hat{S}_t^*}{\hat{S}_t^*} = \sigma d\tilde{Z}.
\]

The asset price \( S_t \) under the equivalent risk neutral measure \( Q \) becomes
\[
\frac{dS_t}{S_t} = (r - q) dt + \sigma d\tilde{Z}.
\]
Hence, the risk neutral drift rate of \( S_t \) is \( r - q \).

**Analogy with foreign currency options**

The continuous yield model is also applicable to *options on foreign currencies* where the continuous dividend yield can be considered as the yield due to the interest earned by the foreign currency at the foreign interest rate \( r_f \).
Call and put price formulas

The price of a European call option on a continuous dividend paying asset can be obtained by changing $S$ to $Se^{-q\tau}$ in the price formula.

This rule of transformation is justified since the drift rate of the dividend yield paying asset under the risk neutral measure is $r-q$. Now, the European call price formula with continuous dividend yield $q$ is found to be

$$c = Se^{-q\tau}N(\hat{d}_1) - Xe^{-r\tau}N(\hat{d}_2),$$

where

$$\hat{d}_1 = \frac{\ln \frac{S}{X} + (r-q + \frac{\sigma^2}{2})\tau}{\sigma \sqrt{\tau}}, \quad \hat{d}_2 = \hat{d}_1 - \sigma \sqrt{\tau}.$$
Similarly, the European put formula with continuous dividend yield $q$ can be deduced from the Black-Scholes put price formula to be

$$p = X e^{-r\tau} N(-\hat{d}_2) - S e^{-q\tau} N(-\hat{d}_1).$$

The new put and call prices satisfy the put-call parity relation

$$p = c - S e^{-q\tau} + X e^{-r\tau}.$$

Furthermore, the following put-call symmetry relation can also be deduced from the above call and put price formulas

$$c(S, \tau; X, r, q) = p(X, \tau; S, q, r),$$
• The put price formula can be obtained from the corresponding call price formula by interchanging $S$ with $X$ and $r$ with $q$ in the formula. Recall that a call option entitles its holder the right to exchange the riskless asset for the risky asset, and vice versa for a put option. The dividend yield earned from the risky asset is $q$ while that from the riskless asset is $r$.

• If we interchange the roles of the riskless asset and risky asset in a call option, the call becomes a put option, thus giving the justification for the put-call symmetry relation.
Time dependent parameters

Suppose the model parameters become deterministic functions of time, the Black-Scholes equation has to be modified as follows
\[
\frac{\partial V}{\partial \tau} = \frac{\sigma^2(\tau)}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \left[ r(\tau) - q(\tau) \right] S \frac{\partial V}{\partial S} - r(\tau) V, \quad 0 < S < \infty, \quad \tau > 0,
\]
where \( V \) is the price of the derivative security.

When we apply the following transformations: \( y = \ln S \) and \( w = e^{\int_0^\tau r(u) du} V \), then
\[
\frac{\partial w}{\partial \tau} = \frac{\sigma^2(\tau)}{2} \frac{\partial^2 w}{\partial y^2} + \left[ r(\tau) - q(\tau) - \frac{\sigma^2(\tau)}{2} \right] \frac{\partial w}{\partial y}.
\]
Consider the following form of the fundamental solution
\[
f(y, \tau) = \frac{1}{\sqrt{2\pi s(\tau)}} \exp \left( - \frac{[y + e(\tau)]^2}{2s(\tau)} \right),
\]
it can be shown that \( f(y, \tau) \) satisfies the parabolic equation

\[
\frac{\partial f}{\partial \tau} = \frac{1}{2} s'(\tau) \frac{\partial^2 f}{\partial y^2} + e'(\tau) \frac{\partial f}{\partial y}.
\]

Suppose we let

\[
s(\tau) = \int_0^\tau \sigma^2(u) \, du
\]

\[
e(\tau) = \int_0^\tau \left[ r(u) - q(u) \right] \, du - \frac{s(\tau)}{2},
\]

one can deduce that the fundamental solution is given by

\[
\phi(y, \tau) = \frac{1}{\sqrt{2\pi \int_0^\tau \sigma^2(u) \, du}} \exp \left( \frac{- \left\{ y + \int_0^\tau \left[ r(u) - q(u) - \frac{\sigma^2(u)}{2} \right] \, du \right\}^2}{2 \int_0^\tau \sigma^2(u) \, du} \right).
\]

Given the initial condition \( w(y, 0) \), the solution can be expressed as

\[
w(y, \tau) = \int_{-\infty}^{\infty} w(\xi, 0) \phi(y - \xi, \tau) \, d\xi.
\]
Note that the time dependency of the coefficients \( r(\tau), q(\tau) \) and \( \sigma^2(\tau) \) will not affect the spatial integration with respect to \( \xi \). We make the following substitutions in the option price formulas:

\[
\begin{align*}
    r & \text{ is replaced by } \frac{1}{\tau} \int_0^\tau r(u) \, du, \\
    q & \text{ is replaced by } \frac{1}{\tau} \int_0^\tau q(u) \, du, \\
    \sigma^2 & \text{ is replaced by } \frac{1}{\tau} \int_0^\tau \sigma^2(u) \, du.
\end{align*}
\]

For example, the European call price formula is modified as follows:

\[
c = S e^{-\int_0^\tau q(u) \, du} N(\tilde{d}_1) - X e^{-\int_0^\tau r(u) \, du} N(\tilde{d}_2)
\]

where

\[
\begin{align*}
    \tilde{d}_1 &= \ln \frac{S}{X} + \int_0^\tau [r(u) - q(u) + \frac{\sigma^2(u)}{2}] \, du, \\
    \tilde{d}_2 &= \tilde{d}_1 - \sqrt{\int_0^\tau \sigma^2(u) \, du}.
\end{align*}
\]
**Implied volatilities**

The only unobservable parameter in the Black-Scholes formulas is the volatility value, $\sigma$. By inputting an estimated volatility value, we obtain the option price. Conversely, given the market price of an option, we can back out the corresponding *Black-Scholes implied volatility*.

- Several implied volatility values obtained simultaneously from different options (varying strikes and maturities) on the same underlying asset provide the market view about the volatility of the stochastic movement of the asset price.
- Given the market prices of European call options with different maturities (all have the strike prices of 105, current asset price is 106.25 and short-term interest rate over the period is flat at 5.6%).

<table>
<thead>
<tr>
<th>maturity</th>
<th>1-month</th>
<th>3-month</th>
<th>7-month</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>3.50</td>
<td>5.76</td>
<td>7.97</td>
</tr>
<tr>
<td>Implied volatility</td>
<td>21.2%</td>
<td>30.5%</td>
<td>19.4%</td>
</tr>
</tbody>
</table>
**Time dependent volatility**

The Black-Scholes formulas remain valid for time dependent volatility except that \( \sqrt{\frac{1}{T-t} \int_t^T \sigma(\tau)^2 d\tau} \) is used to replace \( \sigma \).

How to obtain \( \sigma(t) \) given the implied volatility measured at time \( t^* \) of a European option expiring at time \( t \). Now

\[
\sigma_{imp}(t^*, t) = \sqrt{\frac{1}{t-t^*} \int_{t^*}^t \sigma(\tau)^2 d\tau}
\]

so that

\[
\int_{t^*}^t \sigma(\tau)^2 d\tau = \sigma_{imp}^2(t^*, t)(t - t^*).
\]

Differentiate with respect to \( t \), we obtain

\[
\sigma(t) = \sqrt{\sigma_{imp}(t^*, t)^2 + 2(t - t^*)\sigma_{imp}(t^*, t) \frac{\partial \sigma_{imp}(t^*, t)}{\partial t}}.
\]
Practically, we do not have a continuous differentiable implied volatility function $\sigma_{imp}(t^*, t)$, but rather implied volatilities are available at discrete instants $t_i$. Suppose we assume $\sigma(t)$ to be piecewise constant over $(t_{i-1}, t_i)$, then

\[
(t_i - t^*)\sigma_{imp}^2(t^*, t_i) - (t_{i-1} - t^*)\sigma_{imp}^2(t^*, t_{i-1}) = \int_{t_{i-1}}^{t_i} \sigma^2(\tau) d\tau = \sigma^2(t)(t_i - t_{i-1}), \quad t_{i-1} < t < t_i,
\]

\[
\sigma(t) = \sqrt{\frac{(t_i - t^*)\sigma_{imp}^2(t^*, t_{i-1}) - (t_{i-1} - t^*)\sigma_{imp}^2(t^*, t_{i-1})}{t_i - t_{i-1}}}, \quad t_{i-1} < t < t_i.
\]
Quanto-prewashing techniques

1. Consider two assets whose dynamics follow the lognormal processes

\[
\frac{df}{f} = \mu_f dt + \sigma_f dZ_f \\
\frac{dg}{g} = \mu_g dt + \sigma_g dZ_g.
\]

By the Ito Lemma

\[
\frac{d(fg)}{fg} = (\mu_f + \mu_g + \rho \sigma_f \sigma_g) dt + \sigma dZ
\]

where \( \sigma^2 = \sigma_f^2 + \sigma_g^2 + 2\rho \sigma_f \sigma_g; \)

\[
\frac{d(f/g)}{f/g} = (\mu_f - \mu_g - \rho \sigma_f \sigma_g + \sigma_g^2) dt + \tilde{\sigma} dZ
\]

where \( \tilde{\sigma}^2 = \sigma_f^2 + \sigma_g^2 - 2\rho \sigma_f \sigma_g. \)
**Proof**

\[
\begin{align*}
\text{d}(fg) &= f\,d g + g\,d f + \underbrace{\rho \sigma_f \sigma_g fg\,dt}_{\text{arising from } df dg \text{ and }
\text{observing } dZ_f dZ_g = \rho\,dt} \\
\frac{d(fg)}{fg} &= \frac{d g}{g} + \frac{d f}{f} + \rho \sigma_f \sigma_g dt \\
&= (\mu_f + \mu_g + \rho \sigma_f \sigma_g)\,dt + \sigma_f\,dZ_f + \sigma_g\,dZ_g. \quad (1)
\end{align*}
\]

Observe that the sum of two Brownian processes remains to be Brownian. Recall the formula

\[
\text{VAR}(X + Y) = \text{VAR}(X) + \text{VAR}(Y) + 2\text{COV}(X, Y).
\]

Hence, the sum of \(\sigma_f\,dZ_f + \sigma_g\,dZ_g\) can be expressed as \(\sigma\,dZ\), where

\[
\sigma^2 = \sigma_f^2 + \sigma_g^2 + 2\rho \sigma_f \sigma_g.
\]
\[ d \left( \frac{1}{g} \right) = -\frac{dg}{g^2} + \frac{2}{g^3} \frac{dg^2}{2}; \]  
(2)

\[ \frac{d}{1} \left( \frac{1}{g} \right) = -\mu_g dt + \sigma_g^2 dt - \sigma_g dZ_g. \]  
(3)

Replacing \( g \) by \( 1/g \) in formula (1), we obtain
\[ \frac{d(f/g)}{f/g} = (\mu_f - \mu_g - \rho\sigma_f\sigma_g + \sigma^2_g) dt + \sigma_f dZ_f - \sigma_g dZ_g. \]  
(4)

Recall the formula: \( \text{VAR}(X - Y) = \text{VAR}(X) + \text{VAR}(Y) - 2\text{COV}(X, Y) \). The sum of \( \sigma_f dZ_f - \sigma_g dZ_y \) can be expressed as \( \tilde{\sigma} dZ \), where
\[ \tilde{\sigma}^2 = \sigma_f^2 + \sigma_g^2 - 2\rho\sigma_f\sigma_g. \]
2. \( S \) = foreign asset price in foreign currency
   \( F \) = exchange rate
   = domestic currency price of one unit of foreign currency
   \( S^* = FS \) = foreign asset price in domestic currency
   \( q \) = dividend yield of the asset.

Under the domestic risk neutral measure \( Q_d \), the risk neutral drift rate of \( S^* \) and \( F \) are

\[
\delta_{S^*}^d = r_d - q \quad \text{and} \quad \delta_F^d = r_d - r_f.
\]

Under the foreign \( Q_f \), the risk neutral drift rate for \( S \) and \( 1/F \) are

\[
\delta_S^f = r_f - q \quad \text{and} \quad \delta_{1/F}^f = r_f - r_d.
\]
Quanto prewashing is to find $\delta^d_S$, that is, the risk neutral drift rate of the price of the asset in foreign currency under $Q_d$.

Recall the formula:

$$\delta^d_S = \delta^d_{FS} = \delta^d_F + \delta^d_S + \rho \sigma_F \sigma_S$$

where the dynamics of $S$ and $F$ under $Q_d$ are

$$\frac{dS}{S} = \delta^d_S dt + \sigma_S dZ^d_S$$
$$\frac{dF}{F} = \delta^d_F dt + \sigma_F dZ^d_F,$$

where $Z^d_S$ and $Z^d_F$ are $Q_d$-Brownian process.
We obtain
\[ \delta^d_S = \delta^d_{S^*} - \delta^d_F - \rho \sigma_F \sigma_S \]
\[ = (r_d - q) - (r_d - r_f) - \rho \sigma_F \sigma_S \]
\[ = (r_f - q) - \rho \sigma_F \sigma_S. \]

Comparing with \( \delta^f_S = r_f - q \) and \( \delta^d_S \), there is an extra term \(-\rho \sigma_F \sigma_S \).

The risk neutral drift rate of the asset is changed by the amount \(-\rho \sigma_F \sigma_S \) when the risk neutral measure is changed from the foreign currency world to the domestic currency world.
Siegel’s paradox

Given that the price dynamics of $F$ under $Q_d$ is

$$\frac{dF}{F} = (r_d - r_f) \, dt + \sigma_F \, dZ_d,$$

then the process for $1/F$ is

$$\frac{d(1/F)}{1/F} = (r_f - r_d + \sigma_F^2) \, dt - \sigma_F \, dZ_d.$$  

This is seen as a puzzle to many people since the risk neutral drift rate for $1/F$ should be $r_f - r_d$ instead of $r_f - r_d + \sigma_F^2$.

We observe directly from the above SDE’s that

$$\sigma_F = \sigma_{1/F} \quad \text{and} \quad \rho_{F,1/F} = -1.$$
An interesting application of Siegel’s paradox

Suppose the terminal payoff of an exchange rate option is \( F_T \mathbbm{1}_{\{F_T > K\}} \). Let \( V^d(F, t) \) denote the value of the option in the domestic currency world. Define

\[
V^f(F_t, t) = \frac{V^d(F_t, t)}{F_t},
\]

so that the terminal payoff of the exchange rate option in foreign currency world is \( \mathbbm{1}_{\{F_T > K\}} \). Now

\[
V^f(F, t) = e^{-r_f(T-t)}E_t^{Q_f}[\mathbbm{1}_{\{F_T > K\}}|F_t = F].
\]
From $\delta_{1/F}^d = \delta_{1/F}^f + \sigma_F^2$ and observing $\sigma_F = \sigma_{1/F}$, we deduce that

$$\delta_{F}^f = \delta_{F}^d + \sigma_F^2.$$  

This is easily seen if we interchange the foreign and domestic currency worlds. We obtain

$$V_d(F, t) = F V_f(F, t) = e^{-r_f(T-t)} F N(d)$$

where

$$d = \frac{\ln \frac{F}{K} + \left( \frac{\delta_{F}^f - \sigma_F^2}{2} \right) \tau}{\sigma \sqrt{\tau}} = \frac{\ln \frac{F}{K} + \left( r_d - r_f + \frac{\sigma_F^2}{2} \right) \tau}{\sigma \sqrt{\tau}}.$$
**Foreign exchange options**

Under the domestic risk neutral measure $Q^d$, the exchange rate process follows

$$\frac{dF}{F} = (r_d - r_f)dt + \sigma_F dZ_F.$$

Suppose the terminal payoff is $\max(F_T - X_d, 0)$, then the price of the exchange rate call option is

$$V(F, \tau) = Fe^{-r_f \tau} N(d_1) - X_d e^{-r_d \tau} N(d_2),$$

where

$$d_1 = \frac{\ln \frac{F}{X_d} + \left(r_d - r_f + \frac{\sigma_F^2}{2}\right) \tau}{\sigma_F \sqrt{\tau}}, \quad d_2 = d_1 - \sigma_F \sqrt{\tau}.$$
Equity options with exchange rate risk exposure

Quanto options are contingent claims whose payoff is determined by a financial price or index in one currency but the actual payout is done in another currency.

1. Foreign equity call struck in foreign currency

\[ c_1(S_T, F, 0) = F_T \max(S_T - X_f, 0). \]

2. Foreign equity call struck in domestic currency

\[ c_2(S_T, F, 0) = \max(F_T S_T - X_d, 0). \]

3. Foreign exchange rate foreign equity call

\[ c_3(S_T, F, 0) = F_0 \max(S_T - X_f, 0). \]
Under the domestic risk neutral measure $Q_d$

\[
\begin{align*}
\frac{dS}{S} &= \delta_S^d dt + \sigma_S dZ_S \\
\frac{dF}{F} &= \delta_F^d dt + \sigma_F dZ_F \\
S^* &= FS = \text{asset price in domestic currency} \\
\frac{dS^*}{S^*} &= \delta_{S*}^d dt + \sigma_{S*} dZ_{S*}.
\end{align*}
\]

where $Z_S, Z_F$, and $Z_{S*}$ are all $Q_d$-Brownian processes.

By Ito's lemma,

\[
\begin{align*}
\delta_{S*}^d &= \delta_S^d + \delta_F^d + \rho_{SF}\sigma_S\sigma_F \\
\sigma_{S*}^2 &= \sigma_S^2 + \sigma_F^2 + 2\rho_{SF}\sigma_S\sigma_F.
\end{align*}
\]

Under the risk neutral measures,

\[
\begin{align*}
\delta_{S*}^d &= r_d - q, \quad \delta_F^d = r_d - r_f, \quad \delta_S^f = r_f - q, \\
\delta_S^d &= \delta_{S*}^d - \delta_F^d - \rho_{SF}\sigma_S\sigma_F = r_f - q - \rho_{SF}\sigma_S\sigma_F = \delta_S^f - \rho_{SF}\sigma_S\sigma_F.
\end{align*}
\]
1. Define

\[ c_1(S, F, \tau)/F = \hat{c}_1(S, \tau) \text{ so that } \hat{c}_1(S, 0) = \max(S - X_f, 0). \]

This call option behaves like the usual vanilla call option in the foreign currency world so that

\[ \hat{c}_1(S, 0) = S e^{-q\tau} N(d_1^{(1)}) - X_f e^{-r_f \tau} N(d_2^{(1)}) \]

where

\[ d_1^{(1)} = \frac{\ln \frac{S}{X_f} + \left( \delta^f_S + \frac{\sigma_S^2}{2} \right) \tau}{\sigma_S \sqrt{\tau}}, \quad d_2^{(1)} = d_1^{(1)} - \sigma_S \sqrt{\tau}. \]
2. 
\[ c_2(S, F, 0) = \max(S_T^* - X_d, 0) \]
\[ c_2(S, F, \tau) = S^* e^{-q\tau} N(d_1^{(2)}) - X_d e^{-r_d\tau} N(d_2^{(2)}) \]

where

\[ d_1^{(2)} = \ln \frac{S^*}{X_d} + \left( \delta_{S^*} d + \frac{\sigma_{S^*}^2}{2} \right) \tau, \quad d_2^{(2)} = d_1^{(2)} - \sigma_{S^*}^* \sqrt{\tau}, \]
\[ \sigma_{S^*}^2 = \sigma_S^2 + \sigma_F^2 + 2\rho_{SF} \sigma_S \sigma_F. \]
3. For \( c_3(S, 0) = F_0 \max(S_T - X_f, 0) \), the payoff is denominated in the domestic currency world, so the risk neutral drift rate is of the stock price \( \delta^d_S \). The call price is

\[
c_3(S, \tau) = F_0 e^{-r_d \tau} [S e^{\delta^d_S \tau} N(d_{1(3)}) - X_f N(d_{2(3)})]
\]

where

\[
d_{1(3)} = \ln \frac{S}{X_f} + \left( \delta^d_S + \frac{\sigma^2_S}{2} \right) \tau, \quad d_{2(3)} = d_{1(3)} - \sigma_S \sqrt{\tau}.
\]

- The price formula does not depend on the exchange rate \( F \) since the exchange rate has been chosen to be the fixed value \( F_0 \).
- The currency exposure is reflected through the dependence on \( \sigma_F \) and correlation coefficient \( \rho_{SF} \).
Digital quanto option relating 3 currency worlds

\[ F_{S\backslash U} = \text{SGD currency price of one unit of USD currency} \]

\[ F_{H\backslash S} = \text{HKD currency price of one unit of SGD currency} \]

- Digital quanto option payoff: pay one HKD if \( F_{S\backslash U} \) is above some strike level \( K \).

- We may interest \( F_{S\backslash U} \) as the price process of a tradeable asset in SGD. The dynamics is governed by

\[
\frac{dF_{S\backslash U}}{F_{S\backslash U}} = (r_{SGD} - r_{USD}) \, dt + \sigma_{F_{S\backslash U}} \, dZ^S_{F_{S\backslash U}}.
\]
• Given $\delta_{FS\text{\|U}}^S = r_{SGD} - r_{USD}$, how to find $\delta_{FS\text{\|U}}^H$, which is the risk neutral drift rate of the SGD asset denominated in Hong Kong dollar?

• By the quanto-prewashing technique

$$\delta_{FS\text{\|U}}^H = \delta_{FS\text{\|U}}^S - \rho \sigma_{FS\text{\|U}} \sigma_{FH\text{\|S}}.$$

• Digital option value $=$ $E_t^{QH} \left[ \mathbf{1}_{\{FS\text{\|U}>K\}} \right] = N(d)$

where

$$d = \frac{\ln \frac{FS\text{\|U}}{K} + \left( \delta_{FS\text{\|U}}^H - \frac{\sigma_{FS\text{\|U}}^2}{2} \right) \tau}{\sigma_{FS\text{\|U}} \sqrt{\tau}}.$$
Exchange options

Exchange asset 2 for asset 1 so that the terminal payoff is

\[ V(S_1, S_2, 0) = \max(S_1 - S_2, 0). \]

Assume

\[
\frac{dS_1}{S_1} = \delta_{S_1} \, dt + \sigma_1 \, dZ_1, \quad \delta_{S_1} = r - q_1,
\]

\[
\frac{dS_2}{S_2} = \delta_{S_2} \, dt + \sigma_2 \, dZ_2, \quad \delta_{S_2} = r - q_2.
\]

Define \( S = \frac{S_1}{S_2}, \sigma^2_S = \sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2 \). Write \( \frac{V(S_1, S_2, 0)}{S_2} = \max \left( \frac{S_1}{S_2} - 1, 0 \right) \).

\[ V(S_1, S_2, \tau) = e^{-r\tau} \left[ S_1 e^{\delta_{S_1}\tau} N(d_1) - S_2 e^{\delta_{S_2}\tau} N(d_2) \right], \]

where

\[
d_1 = \frac{\ln \left( \frac{S_1}{S_2} \right) + \left( \delta_{S_1} - \delta_{S_2} \right) + \frac{\sigma^2_S}{2} \tau}{\hat{\sigma}_S \sqrt{\tau}}, \quad d_2 = d_1 - \hat{\sigma}_S \sqrt{\tau}.
\]
Hints on the proof of the price formula

Under the risk neutral measure

\[
\frac{dS}{S} = (\delta S_1 - \delta S_2 - \rho_{12} \sigma_1 \sigma_2 + \sigma_2^2) \, dt + \sigma_1 \, dZ_1 - \sigma_2 \, dZ_2.
\]

It is convenient to use \( S_2(t)e^{q_2 t} \) as numeraire, where

\[
S_2(t) = S_2(0)e^{(\delta S_2 - \frac{\sigma_2^2}{2})t + \sigma_2 Z_2(t)}
\]

or

\[
\frac{S_2(t)e^{q_2 t}}{S_2(0)}e^{-rt} = e^{-\frac{\sigma_2^2}{2} t + \sigma_2 Z_2(t)}.
\]

We can take \( \gamma = -\sigma_2 \) in Girsanov Theorem so that

\[
\frac{S_2(t)e^{q_2 t}}{S_2(0)}e^{-rt} = e^{-\frac{1}{2} \sigma_2^2 t + \sigma_2 Z_2(t)} = \frac{dQ^*}{dQ}.
\]
We then have \( d\hat{Z}_2 = dZ_2 - \sigma_2 dt \) where \( d\hat{Z}_2 \) is under \( Q^* \). In a similar manner, we obtain

\[
d\hat{Z}_1 = dZ_1 - \rho_{12}\sigma_2 dt.
\]

Putting everything together,

\[
\frac{dS}{S} = (\delta_{S_1} - \delta_{S_2}) dt + (\sigma_1 d\hat{Z}_1 - \sigma_2 d\hat{Z}_2)
\]

and

\[
\sigma_1 d\hat{Z}_1 - \sigma_2 d\hat{Z}_2 = \hat{\sigma} d\hat{Z}
\]

where

\[
\hat{\sigma}^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2.
\]
Use of the exchange option formula to price quanto options

Consider

\[ C_2(S, F, 0) = F \max(S - X, 0) = \max(S^* - XF, 0) \]
\[ C_4(S, F, 0) = S \max(F - X, 0) = \max(S^* - XS, 0) \]

where \( S^* = FS \). Both can be considered as exchange options.

Though an exchange option appears to be a two-state option, it can be reduced to an one-state pricing model when the similarity variable \( S = \frac{S_1}{S_2} \) is defined. Similarly, the two-state quanto options can be reduced to one-state pricing models.
For valuation of $C_4(S, F, \tau)$, we consider the similarity variable $\frac{S^*}{S} = F$. Note that

$$\delta^d_{S^*} = \delta^d_S + \delta^d_F + \rho_{SF} \sigma_S \sigma_F, \quad \hat{\sigma} = \sigma_F,$$

and the corresponding difference in the risk neutral drift rates in $Q_d$ is

$$\delta^d_{S^*} - \delta^d_S = \delta^d_F + \rho_{SF} \sigma_S \sigma_F = r_d - r_f + \rho_{SF} \sigma_S \sigma_F.$$