Numerical Algorithms for Pricing Discrete Variance and Volatility Derivatives under Time-changed Lévy Processes

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Abstract
We propose robust numerical algorithms for pricing discrete variance options and volatility swaps under general time-changed Lévy processes. Since analytic pricing formulas of these derivatives are not available, some of the earlier pricing methods use the quadratic variation approximation for the discrete realized variance. While this approximation works quite well for long-maturity options on discrete realized variance, numerical accuracy deteriorates for options with low frequency of monitoring and/or short maturity. To circumvent these shortcomings, we construct numerical algorithms that rely on the computation of the moment generating function of the discrete realized variance under the time-changed Lévy models. We adopt the randomization of the Laplace transform of the discrete log return with a standard normal random variable and develop a recursive quadrature algorithm to compute the moment generating function of the discrete realized variance. Our pricing approach is rather computationally efficient when compared with the Monte Carlo simulation and works particularly well for discrete realized variance and volatility derivatives with low frequency of monitoring and/or short maturity. The pricing properties of various variance and volatility derivatives under various time-changed Lévy processes and the Heston model are also investigated.

Keywords: variance swaps, volatility swaps, variance options, time-changed Lévy processes, discrete sampling

1 Introduction

The discrete realized variance of the log return of a risky stock is defined as the squared log returns of the stock price observed on a set of discrete monitoring dates. Variance swaps are forward contracts on the discrete realized variance of the price process of an underlying stock. They have pure exposure to volatility when used in trading and risk management of volatility when compared to the use of conventional stock options. The pricing of variance and volatility derivatives under various models of the stock price dynamics has been well explored in the literature. The pricing approaches can be divided into two categories. The first category is the non-parametric approach that has no dependence on a particular choice of the asset price model. The fair strike of a variance swap with continuous sampling can be obtained by the notion of replication of a continuum of stock options with varying strikes. This approach has been explained in the works by Neuberger (1994), Demeterfi et al. (1999) and Carr and Lee (2009). The shortcoming of this approach is that option prices are only

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available for a limited number of strikes while replication of the variance swap requires the market prices of options for a full spectrum of strikes.

In the model-dependent pricing approach, the quadratic variation of the continuous realized variance is commonly adopted as an approximation to its discrete counterpart since it provides good analytical tractability by virtue of an explicit representation of the associated characteristic function. Earlier research works that employ the quadratic variation approximation for pricing variance swaps have been performed under the Heston stochastic volatility model (Swishchuk, 2004), the 3/2-model (Carr and Sun, 2007) and the time-changed Lévy processes (Carr et al., 2012). For pricing of options on variance swaps under various stochastic volatility models, one may refer to the papers by Carr et al. (2005) for pure jump processes, Sepp (2008) for an extended Heston model with simultaneous jumps in the underlying and variance processes (SVJJ model), Drimus (2012) and Goard and Mazur (2013) for the 3/2-stochastic volatility model, and Kallsen et al. (2011) for general affine models with jumps. The quadratic variation approximation is seen not to perform well for short-maturity variance derivatives with non-linear payoffs, like the volatility swaps and options on discrete realized variance. Keller-Ressel and Muhle-Karbe (2013) propose an exact pricing formula for variance options under the Lévy models by randomizing the Laplace transform of the discrete log return with a standard normal variate.

Instead of adopting the quadratic variation approximation under the continuous realized variance, the recent papers deal directly with the contractual specification of discrete monitoring of the realized variance. Broadie and Jain (2008) and Zhu and Lian (2011) derive closed form formula for discrete variance swaps under the Heston model. Zheng and Kwok (2014a) consider pricing exotic variance swaps (including the gamma swaps and corridor swaps) under the stochastic volatility model with simultaneous jumps (SVJJ model). Itkin and Carr (2010) employ the forward characteristic function method to price variance swaps under the time-changed Lévy processes with the Cox-Ingersoll-Ross (CIR) and 3/2-processes as the clock rates.

Besides the variance swap products, other researchers have proposed various analytic approximation methods or numerical algorithms to price discretely monitored variance derivatives with non-linear terminal payoff structures. Drimus and Farkas (2010) study the discretization errors on variance options arising from continuous sampling approximation via the central limit theorem. They propose a simplified one-dimensional Monte Carlo simulation method and derive a set of analytic pricing formulas based on the asymptotic distribution. For low frequency of sampling of the realized variance, their approximation method does not perform quite well due to poor approximation by the normal distribution using a few sampling points. Sepp (2012) approximates the discrete realized variance by the continuous realized variance with an additional correction term from a lognormal model in the SVJJ model. This method works well for options that are near-the-money (see also Drimus and Farkas, 2010). Chiarella et al. (2013) approximate the characteristic function of the discrete variance in the Heston model and derive semi-analytic formulas for variance derivatives. Zheng and Kwok (2014b) apply the saddlepoint approximation for pricing options on discrete realized variance. They also propose an enhanced simulation method for pricing the same class of products under a class of stochastic volatility models with jumps. Though their saddlepoint method approximation works well in general, the simulation method does not perform well for short-maturity put options and volatility swaps.

For the literature on the numerical algorithms for pricing discrete variance derivatives, Little and Pant (2001) compute the fair strike of discrete variance swaps under a local volatility model by the finite difference scheme. Windcliff et al. (2006) extend the Little-Pant algorithm by incorporating jumps in the stock price dynamics. Baldeaux (2011) proposes an exact simulation for the 3/2-model with some variance reduction technique by applying
the Lie symmetry analysis. Zheng and Kwok (2014c) employ the Fourier time-stepping algorithm to price variance derivatives under the additive processes. They adopt the CONV method (Lord et al., 2008) in their fast Fourier transform algorithm. Though it is desirable to incorporate stochastic volatility in the underlying stock price dynamics, the direct extension of their numerical scheme to the time-changed Lévy processes or stochastic volatility models is computationally infeasible due to curse of dimensionality.

The recursion algorithms have been widely used to price various kind of options under the Lévy processes, like pricing discretely monitored Asian options by Fusai and Meucci (2008). Similar approach has been used by Sullivan (2000) and Fusai and Recchioni (2007) to price discrete barrier options. Yamazaki (2013) and Umezawa and Yamazaki (2012) employ a similar recursive algorithm to price path-dependent stock options under the time-changed Lévy processes.

In this paper, we extend the recursive algorithms to price discrete variance and volatility derivatives under the time-changed Lévy processes. We employ the randomization formula in a recursive quadrature algorithm to compute the moment generating function of the discrete realized variance. The explicit Laplace transform formula of the activity rate process essentially reduces the additional computational cost that arises from the extra state variable in the time-changed Lévy processes. This key step makes the numerical pricing of discrete variance derivatives under the time-changed Lévy processes computationally feasible. We propose effective recursive algorithms to compute the moment generating function of the discrete variance under the time-changed Lévy models. Our method works particularly well for variance and volatility derivatives with short-maturity and low sampling frequency. The pricing errors are small even with reasonably large number of recursive iterations in the numerical computation. In the first stage of the construction of our proposed algorithms, we assume independence between the underlying Lévy process and the activity rate process. By randomizing the squared terms in the realized variance formula with an independent standard normal variable, the sum of the squared returns can be expressed as linear terms under the expectation of the normal variable. This expectation is iterating on each monitoring dates and can be effectively computed in term of a “transition matrix”. Option prices and the fair strikes of discrete volatility swaps can be obtained through an inverse Laplace transform. To accommodate the leverage effect, we introduce a specific correlated diffusion term into the underlying Lévy process.

This paper is organized as follows. In the next section, we provide a brief description of the time-changed Lévy processes and the activity rate processes. The analytic formulas for the Laplace transform of the joint density of variance and integrated variance are derived. In Section 3, the general framework of our pricing approach is presented. The implementation of our numerical method is outlined in matrix forms. Numerical tests on numerical accuracy of our algorithms are presented in Section 4. Comparative analysis on different parameters are also studied in this section. Finally, we conclude our results in Section 5.

2 Time-changed Lévy processes

Given the stationary increment property of the Lévy process, it is not feasible to generate stochastic volatility. The time-changed Lévy processes consider subordinating the time with a stochastic activity rate process (Carr and Wu, 2004). These time-changed Lévy processes can generate stochastic volatility, and can also incorporate the leverage effect by introducing a negative correlation between the Lévy process and the activity rate process. Furthermore, the reverting feature of the activity rate process (in the form of the CIR process or the 3/2-process) can capture the volatility clustering phenomenon observed in the studies of time
series data in the stock market.

2.1 Lévy processes

We start with a probability space \((\Omega, \mathcal{F}, Q)\) with a complete filtration \(\mathcal{F} = \{\mathcal{F}_t | t \geq 0\}\), where \(Q\) is a risk neutral measure. A real-valued stochastic process \(X_t\) with \(X_0 = 0\) is a Lévy process if it has independent and stationary increments, implying that \(X_u - X_t\) is independent to \(\mathcal{F}_t\) and equals in distribution as as \(X_u-t\) for \(0 \leq t \leq u\). We assume that \(X_t\) is adapted to \(\mathcal{F}_t\), and it has right continuous sample paths with left limits. As an infinite divisible distribution, \(X_t\) admits the following Lévy-Khintchine representation (Bertoin, 1996) of its characteristic function:

\[
\phi_{X_t}(u) = \mathbb{E}[e^{iuX_t}] = e^{-t\psi_X(u)}, \quad t \geq 0, \tag{2.1}
\]

for \(u \in \mathbb{R}\), where the characteristic exponent \(\psi_X(u)\) is given by

\[
\psi_X(u) = -i\mu u + \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R} \backslash \{0\}} (1 - e^{iu x} + iux\mathbb{1}_{|x|<1})\Pi(dx).
\]

The triplet \((\mu, \sigma^2, \Pi)\) is the Lévy characteristic of \(X\), where \(\mu \in \mathbb{R}\) is the constant drift term, \(\sigma \geq 0\) is the constant variance of the continuous component and \(\Pi\) is the Lévy density that represents the arrival rate of jumps of different jump sizes that satisfies the finite quadratic variation:

\[
\int_{\mathbb{R} \backslash \{0\}} (1 \wedge x^2)\Pi(dx) < \infty.
\]

For a pure jump Lévy process, if the integral of the Lévy density is finite, then the sample paths exhibit finite activity, meaning that only a finite number of jumps occur within any finite interval of time. Otherwise, for infinite activity process, the number of jumps is infinite in any fixed time interval. A Lévy process is completely characterized by its triplet. To obtain a wider application of the Lévy process, we extend the domain of the characteristic function to a subset \(\mathcal{D}\) in the complex plane. For \(u \in \mathcal{D} \subset \mathbb{C}\), the expectation in eq. (2.1) is convergent for \(u \in \mathcal{D}\). The computational domain is assumed to lie within \(\mathcal{D}\).

Under the risk neutral measure \(Q\), the discounted stock price process is a martingale. Suppose we model the stock price dynamics by an exponential Lévy model as follows:

\[
S_t = S_0 \exp((r - q)t + X_t).
\]

Here, \(r\) is the constant risk free rate and \(q\) is the continuous dividend yield. In order to satisfy the martingale condition:

\[
\phi_X(-i) = \mathbb{E}[e^{X_t}] = 1, \tag{2.2}
\]

the drift term \(\mu\) has to be determined by

\[
\mu = -\frac{1}{2}\sigma^2 + \int_{\mathbb{R} \backslash \{0\}} (1 - e^{iu x} + iux\mathbb{1}_{|x|<1})\Pi(dx).
\]

Jumps in the asset price processes have been observed in the financial markets (Geman et al., 2001). Within a small time interval, there may be a lot of small jumps in the stock price movement. It may be difficult to distinguish if the contribution comes from the diffusion or jump component [see Aït-Sahalia (2004) for the methods of detecting jumps]. The financial models with compound Poisson jumps exhibit finite activity of the jump component, as in the Merton model (1976) and Kou model (2002). However, the Poisson process is incapable of capturing the high frequency of small jumps. Alternative Lévy type pure jump models
with infinite activity have been proposed. They show better fit to market data, which include
the Variance Gamma (VG) model of Madan and Seneta (1990), the Normal Inverse Gaussian
(NIG) model (Barndorff-Nislsen, 1998), the generalized hyperbolic model (Eberlein, Keller
and Prause, 1998) and CGMY model (Carr, Geman, Madan and Yor, 2002). All these models
have explicit formulas of their Lévy measures and characteristic exponents, so they provide
good tractability in financial modeling. We choose to use the pure jump models of NIG and
VG (Carr et al., 2003) as demonstrative examples in our numerical tests in Section 4. The
Lévy measures and characteristic exponents of the NIG and VG processes are summarized
below.

1. The NIG process can be generated by time changed a Brownian motion with drift by
an independent Gaussian process. The Lévy density of the NIG process is

$$\nu_{NIG}(x) = \frac{\delta \alpha e^{\beta x} K_1(|x|)}{|x|},$$

where $\alpha > 0$, $-\alpha < \beta < \alpha$, $\delta > 0$ and $K_\lambda(x)$ denotes the modified Bessel function
of the third kind with index $\lambda$. The characteristic exponent is given by

$$\psi_{NIG}(u) = \delta (\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2}).$$

2. The VG process can be obtained by subordinating a Brownian motion by an independent gamma variable. It has the following Lévy measure:

$$\nu_{VG}(x) = \begin{cases} 
C \exp(Gx) & x < 0 \\
C \exp(-Mx) & x > 0 
\end{cases} \frac{|x|}{x},$$

where $C, G$ and $M$ are positive real parameters. The characteristic exponent is given
by

$$\psi_{VG}(u) = -C \ln \left( \frac{GM}{GM + (M - G)iu + u^2} \right).$$

### 2.2 Activity rate processes

The Lévy models provide good fit to the market data of option prices for different strikes at
a fixed maturity date. However, such advantage is not maintained across maturity. In lieu
of the stationary increment assumption in the Lévy models, they do not generate random
movement for the volatility. Stochastic changes of time in the Lévy processes are introduced
as a way to accommodate the stochastic volatility fluctuation.

Specifically, we let the random time $T_t$ be a stopping time with respect to $\mathcal{F}_t$. Also, it is
an increasing right continuous process with left hand limit, finite $\mathbb{Q}$-a.s. and goes to infinite
as $t \to \infty$. The time-changed Lévy process $Y_t$ by $T_t$ is defined by

$$Y_t = X_{T_t}.$$

One common approach to model the time-changed process is to let it be the integrated
non-negative instantaneous process $v_t$ as defined by

$$T_t = \int_0^t v_s \, ds.$$
We take $t$ to be the calendar time and consider $T_t$ as the business time at $t$. A busier business day corresponds to a higher instantaneous activity rate $v_t$ and so the corresponding time-changed process generates a higher volatility. Note that the stochastic time changes can be applied separately to the jump component or diffusion component or both, in order to generate stochastic volatility arising from different scenarios (Huang and Wu, 2004; Carr and Wu, 2005). The correlation between the time-changed process and the underlying Lévy process can be embedded into the model to represent the leverage effect in various ways (Carr and Wu, 2004).

To achieve good analytic tractability, the instantaneous activity rate is usually modeled with tractable characteristic function of the transition density, like the affine processes, Ornstein-Uhlenbeck process and $3/2$-process. As a typical example of one-dimensional affine process, the CIR process has been widely employed in interest rate modeling, while the Heston-type stochastic volatility models are employed in view of its analytic characteristic function. On the other hand, the $3/2$-model becomes popular recently for its wide support from empirical studies, though it is less tractable when compared to the CIR process. In our numerical approach, it is necessary to have an explicit formula for the Laplace transform of the joint transition density in $\int_0^t v_s \, ds$. For the CIR process and the $3/2$-process, their corresponding explicit transformed formulas can be readily found.

**CIR process**

The CIR process $v_t$ is a Markovian stochastic process satisfying the following stochastic differential equation (SDE)

$$dv_t = k(\theta - v_t) \, dt + \epsilon \sqrt{v_t} \, dW^v_t,$$

where $W^v_t$ is a Brownian motion and $k, \theta, \epsilon$ are positive constants. The corresponding transition density can be obtained in closed-form as a non-central chi-squared distribution. If the model parameters satisfy the Feller condition $2k\theta \geq \epsilon^2$, $v_t$ can never reach zero and stay positive. Let $p_{CIR}(v, y, \tau; v_0)$ be the joint transition density of $v_\tau$ and $\int_0^\tau v_s \, ds$ from $(v_0, 0)$ at time 0 to $(v, y)$ at time $\tau$. The following lemma gives the Laplace transform of the joint density $p_{CIR}(v, y, \tau; v_0)$ with respect to $y$.

**Lemma 1** For the CIR process defined in eq. (2.3), the Laplace transform $G_{CIR}(v, \tau; v_0, \eta)$ of the joint density $p_{CIR}(v, y, \tau; v_0)$ with respect to $y$ is given by

$$G_{CIR}(v, \tau; v_0, \eta) = \int_0^\infty e^{-\eta y} p_{CIR}(v, y, \tau; v_0) \, dy$$

$$= 2 \exp \left( \epsilon^2 k (v_0 - v) - \frac{(1 + \epsilon^\delta)(v + v_0)\delta \epsilon^2}{\epsilon^\delta - 1} \right) \left[ \frac{e^{-\delta \tau} (\epsilon^\delta - 1)^2 \epsilon^4}{v_0 \delta^2} \right]^{\frac{1}{2}}$$

$$\left[ \frac{\epsilon^\frac{\delta}{2} (k + \delta \tau) \delta}{(\epsilon^\delta - 1)^2} \right]^\beta I_{\beta-1} \left( 4 \sqrt{\frac{e^{\delta \tau} v_0 \delta^2}{(\epsilon^\delta - 1)^2 \epsilon^4}} \right),$$

where

$$\delta = \sqrt{k^2 + 2 \epsilon^2 \eta} \quad \text{and} \quad \beta = \frac{2k\theta}{\epsilon^2}. $$

Note that $I_\nu$ is the modified Bessel function of the first kind of order $\nu$

$$I_\nu(z) = \left( \frac{1}{2} z \right)^\nu \sum_{k=0}^{\infty} \frac{(\frac{1}{2} z^2)^k}{k! (\nu + k + 1)}.$$

The proof of Lemma 1 is presented in Appendix A.
3/2-model

The 3/2-process $v_t$ is also a Markovian stochastic process satisfying the following SDE

$$dv_t = v_t(p - \tilde{q}v_t)dt + \sigma v_t^{3/2}dW_t^v, \quad (2.5)$$

where $W_t^v$ is a Brownian motion and $p, \tilde{q}, \sigma$ are positive constants. Its sample paths exhibit a more volatile structure than that of the CIR process. These two processes are closely related in the sense that the reciprocal of 3/2-process is a CIR process, so the transition density of the 3/2-process is still available in closed form via some appropriate transformation of variables. Here, we let $p^3_{3/2}(v; y, \tau; v_0)$ be the corresponding joint transition density of $v$ and $\int_0^\tau v_s ds$ at time $\tau$ given $v_0$ at time 0 in the 3/2-process, a similar result on the Laplace transform of the density is presented in Lemma 2.

**Lemma 2** For the 3/2-process defined in eq. (2.5), the Laplace transform $G_{3/2}(v, \tau; v_0, \eta)$ of the joint density $p_{3/2}(v, y, \tau; v_0)$ with respect to $y$ is given by (Zheng and Zeng, 2014)

$$G_{3/2}(v, \tau; v_0, \eta) = \int_0^\infty e^{-\eta y}p_{3/2}(v, y, \tau; v_0) dy = \frac{A_r}{C_r v^2} \exp \left( - \frac{A_r v_0 + v}{C_r v_0 v} \right) \left( \frac{A_r v_0}{v} \right)^{1/2 + \frac{A_r}{2C_r v_0}} I_{2c} \left( 2C_r^{-1} \sqrt{\frac{A_r}{v_0}} \right), \quad (2.6)$$

where

$$c = \sqrt{\left( \frac{1}{2} + \frac{\tilde{q}}{\sigma^2} \right)^2 - \frac{2i\eta}{\sigma^2}}, \quad A_r = e^{pr}, \quad C_r = \frac{\sigma^2(e^{pr} - 1)}{2p}.$$

We assume that the stock price process follows a time-changed Lévy process defined as

$$\ln S_t = (r - q)t + X_{T_i} - \varphi(1)T_i,$$

where $\varphi(u) = -\psi(-iu)$ is the cumulant exponent of $X_t$, and $\varphi(1)T_i$ is the convexity adjustment term so that the discounted stock price satisfies the martingale condition. By changing the drift term $\mu$ to become $\mu - \varphi(1)$, we can rewrite the stock price process as

$$\ln S_t = (r - q)t + \tilde{X}_{T_i}. \quad (2.7)$$

The time-changed Lévy process $X_{T_i}$ is taken to be mean adjusted hereafter. By assuming independence between the underlying Lévy process and the activity rate process, we can express the characteristic function of the time-changed Lévy process $Y_t$ as

$$\phi_{Y_t}(u) = \mathbb{E}[e^{iuY_t}] = \mathbb{E}[e^{-\varphi(1)T_i}] = \mathcal{L}_{T_i}(\varphi X_t(u)),$$

where $\mathcal{L}_{T_i}$ denotes the Laplace transform of the time change $T_i$. Carr and Wu (2004) and Huang and Wu (2004) provide the details of the time-changed Lévy models with leverage effect. Firstly, we derive our numerical method by assuming zero correlation between $X_t$ and $T_i$. We then extend our approach to accommodate the leverage effect under some special type of correlation structure.
3 Formulation of the pricing algorithm

We start with the introduction of the product nature of various discrete variance derivatives, including volatility swaps and options on variance. We then show how to perform numerical computation of the Laplace transform of the discrete realized variance of the stock price process under the time-changed Lévy processes.

3.1 Discrete variance and volatility derivatives

We consider the tenor of the realized variance to be $[0, T]$ with monitoring dates $0 = t_0 < t_1 < \cdots < t_N = T$, where $T$ is the maturity date and $N$ is the total number of monitoring dates. We use $V_d$ to denote the discrete realized variance over $[t_0, t_N]$ defined by

$$V_d = \frac{F_A}{N} \sum_{k=1}^{N} \left( \ln \frac{S_{t_k}}{S_{t_{k-1}}} \right)^2,$$

(3.9)

where $F_A$ is the annualized factor. We take $F_A = 12$ for monthly monitoring and $F_A = 52$ for weekly monitoring. Let $\Delta t_k$ denote the time interval between $t_{k-1}$ and $t_k$, $k = 1, 2, \cdots, N$. We assume the time intervals to be uniform and denote the uniform time interval by $\Delta t$.

1. A discrete variance swap is a forward contract that exchanges the discrete realized variance with a fixed strike. The fair strike $K_{var}$ of the variance swap gives zero value of the contract to both counterparties at initiation. It is given by the risk neutral expectation of $V_d$:

$$K_{var} = \mathbb{E}[V_d].$$

(3.10)

2. A discrete volatility swap is a contract similar to the discrete variance swap. The holder can swap the square root of the discrete realized variance for a fixed strike. Its fair strike is given by

$$K_{vol} = \mathbb{E}[\sqrt{V_d}].$$

(3.11)

3. The option on discrete realized variance gives the holder the right, but not the obligation, to exchange the discrete realized variance for a fixed strike. The fair price $V_p$ of a put option with strike $K$ is given by

$$V_p = \mathbb{E}[(K - V_d)^+].$$

(3.12)

Once we have computed the value of a put option, the value of the call option counterpart can be easily obtained by the call-put parity formula. The fair strike formula for the discrete variance swap under time-changed Lévy process has been derived in Itkin and Carr (2010). However, they resort to analytic approximation when they compute the fair values of the options on variance and volatility swaps. As stated in Bühler (2006), such approximation is not quite accurate for short-maturity variance derivatives with small number of monitoring dates. In this paper, we consider numerical pricing of various types of discrete variance derivatives without any analytic approximation.
3.2 Moment generating function of discrete realized variance

To price discrete variance swaps and other variants, we first compute the moment generating function of the discrete realized variance of the stock price. Let

\[ R_k = \ln \frac{S_{t_k}}{S_{t_{k-1}}} = (r - q)\Delta t_k + X_{T_k} - X_{T_{k-1}} \quad \text{and} \quad I_N = \sum_{k=1}^{N} R_k^2, \]

where we write \( X_{T_k} \) as \( X_{T_{k}} \) for simplicity. We write the Laplace transform of \( I_N \) and \( R_k^2 \) as

\[ \Psi_{I_N}(\lambda) = \mathbb{E}[e^{-\lambda I_N}] \quad \text{and} \quad \Phi_k(\lambda) = \mathbb{E}[e^{-\lambda R_k^2}]. \]

Given the independence assumption of log returns, \( \Psi_{I_N}(\lambda) \) can be expressed as

\[ \Psi_{I_N}(\lambda) = \prod_{k=1}^{N} \Phi_k(\lambda). \quad (3.13) \]

We first consider the trivial time-changed process \( T_t = t \), which is the calendar time itself and then investigate the properties of \( \Psi_{I_N}(\lambda) \). By observing the following relation between \( \psi_R \) and \( \psi_X \), where

\[ e^{-\Delta t_k \psi_R(\lambda)} = \mathbb{E}[e^{\lambda R_k}] = e^{\Delta t_k(i\lambda)(r-q)}\mathbb{E}[e^{i\lambda(X_{T_k} - X_{T_{k-1}})}] = e^{\Delta t_k(i\lambda)(r-q)}e^{-\Delta t_k \psi_X(\lambda)}, \quad (3.14) \]

we can deduce that

\[ \psi_R(\lambda) = -i\lambda(r-q) + \psi_X(\lambda). \]

Keller-Ressel and Muhle-Karbe (2013) extend the Laplace transform to the complex half plane. In order to reduce the squared \( X_t \) in the exponential function to a linear term, they establish the following critical relation

\[ \mathbb{E}[e^{-\lambda X_t^2}] = \mathbb{E}[e^{-t(\psi_X(Z\sqrt{\lambda}))}], \quad (3.15) \]

under a mild technical condition, where \( Z \) is an independent standard normal random variable and \( \lambda \) is in the positive half plane \( \{ \lambda \in \mathbb{C} : \text{Re} \lambda > 0 \} \). The expectation term on the left hand side in the above formula is taken with respect to the Lévy process \( X_t \), while the expectation term on the right hand side is taken with respect to the normal variable \( Z \). A variety of financial models, such as the Merton model, VG and NIG models satisfy the above condition. Applying the above relation for the Lévy process \( R_t \), we obtain the following integral representation of \( \Phi_k(\lambda) \):

\[ \Phi_k(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\Delta t_k(\psi_R(x\sqrt{\lambda})) - x^2/2} \, dx \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\Delta t_k(r-q)x\sqrt{\lambda} - \Delta t_k(\psi_X(x\sqrt{\lambda})) - x^2/2} \, dx. \]

To extend the above representation to the time-changed Lévy processes, we let \( (\mathcal{F}_t)_{t \geq 0} \) be the natural filtration generated by the activity rate process \( (\psi_t)_{0 \leq t \leq T} \). Conditioning on \( \mathcal{F}_T \), we have the following relation corresponding to eq. (3.14):

\[ e^{-\Delta t_k \psi_R(\lambda) \mid_{\mathcal{F}_T}} = e^{\Delta t_k(i\lambda)(r-q)}\mathbb{E}[e^{i\lambda(X_{T_k} - X_{T_{k-1}})} \mid_{\mathcal{F}_T}] = e^{\Delta t_k(i\lambda)(r-q)}e^{-(T_k-T_{k-1})\psi_X(\lambda) \mid_{\mathcal{F}_T}}. \quad (3.16) \]

Due to independence of the two processes, this gives

\[ \Delta t_k \psi_R(\lambda) = -i\lambda(r-q)\Delta t_k + \psi_X(\lambda)(T_k - T_{k-1}). \]
We can obtain the conditional Laplace transform under the time-changed Lévy process as
\[
\Phi_{k|\gamma_T}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\Delta t (r-q) x + \frac{\lambda}{2} x^2} e^{\lambda \int_{0}^{t} v_s \, ds} \, dx.
\]

To recover the unconditional Laplace transform of \( \Psi_I(\lambda) \), we take the expectation with respect to the joint distribution of \( \int_{t_{k-1}}^{t_k} v_s \, ds, \, k = 1, 2, \ldots, N \) in eq. (3.13):
\[
\Psi_I(\lambda) = E \left[ \prod_{k=1}^{N} \Phi_{k|\gamma_T}(\lambda) \right]. \tag{3.17}
\]

In view of the Markovian property in the activity rate process, we manage to derive a recursive formula so that the expectation can be evaluated efficiently. We write
\[
f_k(\lambda, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\Delta t (r-q) x + \frac{\lambda}{2} x^2} e^{\lambda \int_{0}^{t_k} v_s \, ds} \, dx, \quad y \geq 0.
\]

The following proposition provides the details of the computational algorithm.

**Proposition 3** Let \( g_k(\lambda, v) \) be the conditional expectation at time \( t_k \) defined as follows
\[
g_k(\lambda, v) = E \left[ g_{k+1}(\lambda, v_{k+1}) f_k(\lambda, \int_{t_{k-1}}^{t_k} v_s \, ds) \mid v_{t_k} = v \right], \quad k = 1, 2, \ldots, N - 1
\]
\[
g_N(\lambda, v) = 1, \quad \text{for all } v.
\]
The unconditional Laplace transform of \( I_N \) is given by
\[
\Psi_I(\lambda) = g_0(\lambda, v_0). \tag{3.18}
\]

To establish the above result, we consider the expectation in eq. (3.17) and apply the law of iterated expectation at each monitoring time \( t_k, \, k = 1, 2, \ldots, N - 1 \). It then becomes
\[
\Psi_I(\lambda) = E \left[ \Phi_1(\lambda) E \left[ \Phi_2(\lambda) \ldots E \left[ \Phi_N(\lambda) \mid v_{N-1} \right] \ldots \mid v_1 \right] \right]. \tag{3.19}
\]
The innermost expectation integral is seen to be \( g_{N-1}(\lambda, v) \). Iterating recursively, the outermost expectation integral \( \Psi_{I_N}(\lambda) \) is then evaluated as \( g_0(\lambda, v) \).

For the class of tractable activity rate processes such as the CIR and 3/2-processes, we let \( p(\tau, v_k, v, y) \) be the joint transition density of \( (v, \int_{t_{k-1}}^{t_k} v_s \, ds) \) from \( (v, 0) \) at time \( t_{k-1} \) to \( (v_k, y) \) at time \( t_k \). We can further simplify the computation of \( g_k(\lambda, v) \) by using the explicit Laplace transform formulas \( G(\tau, v_k, v, \eta) \) in Lemmas 1 and 2 as follows:
\[
g_k(\lambda, v) = \int_{0}^{\infty} \int_{0}^{\infty} \frac{g_{k+1}(\lambda, v_{k+1})}{\sqrt{2\pi}} \left[ \int_{-\infty}^{\infty} e^{\frac{(r-q) \Delta t k x + \lambda}{2} x^2} e^{\lambda \int_{0}^{t_k} v_s \, ds} \, dx \right] p(v_k, y, \Delta t_k; v) \, dy \, dv_k
\]
\[
= \int_{0}^{\infty} \frac{g_{k+1}(\lambda, v_{k+1})}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{(r-q) \Delta t k x + \lambda}{2} x^2} G(v_k, \Delta t_k; v, \psi_{X}(x \sqrt{2 \lambda})) \, dx \, dv_k. \tag{3.20}
\]
Note that the negative square term of \( x \) in the exponential function provides a fast decaying rate in the computation of the inner integral with respect to \( x \), so a small truncation value for \( x \) would suffice in practical numerical calculations.
3.3 Numerical implementation

It is convenient to rewrite our computational formula in a matrix form so that the recursive iteration can be represented as matrix multiplication. We choose the \( v \)-grid as a \( m \)-point vector \( v = \{v_i\}_{i=1}^m \in \mathbb{R}^m \). We now discretize eq. (3.20) as follows

\[
g_k(\lambda, v_i) = \sum_{j=1}^m w_j g_{k+1}(\lambda, v_j) \left[ \int_{-\infty}^{\infty} e^{(r-q)\Delta t_k \sqrt{2}x - x^2/2} G(v_j, \Delta t_k; v_i; \psi_X(x \sqrt{2\lambda})) \, dx \right], \tag{3.21}
\]

where \( w = \{w_j\}_{j=1}^m \in \mathbb{R}^m \) is a weight vector depending on the discretization. Various numerical integration methods, like the trapezoidal rule or Gauss quadrature rule, can be implemented in our numerical method by choosing a combination of grid points of \( v \) and \( w \). Since we assume uniform time interval \( \Delta t \), the integral in the above equation can be considered as a stationary transition matrix \( H \) whose rows are \( v_i \) and columns are \( v_j, i, j = 1, 2, \ldots, m \). Specifically, for each \( \lambda \), we define a matrix \( H(\lambda) \) whose entry at \( i \)-th row and \( j \)-th column is given by

\[
h_{i,j}(\lambda) = \int_{-\infty}^{\infty} e^{(r-q)\Delta t \sqrt{2}x - x^2/2} G(v_j, \Delta t; v_i; \psi_X(x \sqrt{2\lambda})) \, dx. \tag{3.22}
\]

The recursive algorithm can be presented succinctly in matrix notation as follows

\[
g_k(\lambda, v) = H(\lambda) \text{diag}(w) g_{k+1}(\lambda, v),
\]

with \( g_N(\lambda, v) = (1, \ldots, 1)^T \in \mathbb{R}^m \), where \( \text{diag}(w) \) denotes the diagonal matrix whose diagonal elements are given by the vector \( w \). This formula essentially reduces the iterations to a \( N \)-folded matrix multiplication. Therefore, the Laplace transform \( \Psi_{I_N}(\lambda) \) can be expressed as

\[
g_0(\lambda, v) = [H(\lambda) \text{diag}(w)]^N (1, \ldots, 1)^T. \tag{3.23}
\]

When the initial value \( v_0 \) of the activity rate process is not chosen as a nodal point, interpolation can be applied to obtain \( g_0(\lambda, v_0) \).

3.4 Option pricing formulas

Given the Laplace transform of the discrete realized variance, variance option prices and fair strike of volatility swaps can be derived by using the Laplace inversion formula. Various research papers have discussed different numerical schemes to evaluate the inverse Laplace transform (Kessel-Ressel and Muhle-Karbe, 2013). The transform method in pricing variance options under our numerical method is demonstrated below.

The fair value of a put option on realized variance defined in eq. (3.12) with strike \( K \) is given by the following formula:

\[
\mathbb{E}[(K - V_0)^+] = \frac{F_A}{N} \mathbb{E}[(K' - I_N)^+]
\]

\[
= \frac{F_A}{N} \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{K'(\lambda_r + i\lambda)}}{(\lambda_r + i\lambda)^2} \Psi_{I_N}(\lambda_r + i\lambda) \right] \, d\lambda
\]

\[
= \frac{F_A}{N} \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{K'(\lambda_r + i\lambda)}}{(\lambda_r + i\lambda)^2} g_0(\lambda_r + i\lambda, v_0) \right] \, d\lambda
\]

\[
\approx \frac{F_A}{N} \frac{1}{\pi} \sum_{k=1}^M \omega_k \text{Re} \left[ \frac{e^{K'(\lambda_r + i\lambda_k)}}{(\lambda_r + i\lambda_k)^2} g_0(\lambda_r + i\lambda_k, v_0) \right], \tag{3.24}
\]
where \( K' = K N / F_A \), \( \{ \omega_k \}_{k=1}^M \) is the weight vector of the discretization, \( \{ \lambda_k \}_{k=1}^M \) are the grid points, \( \lambda_r \) is a fixed parameter in the convergence region of \( \Psi_{I_N} \).

By using the following integral representation of the expectation of square root of a random variable (Gatheral, 2006), where

\[
E[\sqrt{X}] = \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{1 - E[e^{-\lambda X}]}{\lambda^{3/2}} \, d\lambda,
\]

the discrete volatility swaps can be priced in a similar manner. The fair strike of the discrete volatility swap is given by

\[
E[\sqrt{V_d}] = \sqrt{\frac{F_A}{N}} \int_{0}^{\infty} \frac{1 - \psi_{I_N}(\lambda)}{\lambda^{3/2}} \, d\lambda
\]

\[
= \frac{1}{2\sqrt{N\pi}} \int_{0}^{\infty} \frac{1 - \lambda_0(\lambda,v_0)}{\lambda^{3/2}} \, d\lambda
\]

\[
\approx \frac{1}{2\sqrt{N\pi}} \sum_{k=1}^{M'} \omega_k' \left[ \frac{1 - \lambda_0(\lambda_k,v_0)}{\lambda_k^{3/2}} \right],
\]

(3.26)

where \( \{ \omega_k' \}_{k=1}^{M'} \) is the weight vector of the discretization. As a remark, some earlier works (Swishchuk, 2004; Brockhaus and Long, 2000) use the convexity adjustment formula

\[
E[\sqrt{V_d}] \approx \sqrt{E[V_d]} - \frac{\text{Var}(V_d)}{8E[V_d]^{3/2}}
\]

(3.27)

to recover the fair strike value of volatility swaps from that of a variance swap straightforwardly. Though the implementation of the convexity adjustment is simple and convenient, Broadie and Jain (2008) show that the convexity adjustment formula works poorly in the Heston model, and even worse in models with jumps. Using eq. (3.26), instead of adopting the convexity adjustment, the fair strike value can be evaluated more accurately by taking a sufficient number of grid points in a larger truncation domain.

### 3.5 Leverage effect

The previous numerical algorithm relies on the assumption of zero leverage in the model. Hence we can apply the randomization by a normal variable and obtain the closed form formula for the time-changed joint density. In general, it cannot be extended to include any kind of leverage effect correlation between the underlying Lévy process and the activity rate process. However, we can introduce the correlation by adding a correlated diffusion term to the asset price process (Kallsen et al., 2011). Specifically, we modify the stock price dynamics in eq. (2.7) to become

\[
\ln \frac{S_t}{S_0} = (r - q) t + X_{T_t} + \rho \int_0^t \sqrt{v_s} \, dW_s^v,
\]

(3.28)

where \( W_s^v \) is the same Brownian motion that drives the activity process \( v_t \) in eq. (2.3) or eq. (2.5). The drift term of the original underlying Lévy process now becomes \( \mu - \varphi(1) - \rho^2/2 \) in order to maintain the martingale property under the modified dynamics.
To explore the types of activity processes that give a similar expression as in eq. (3.20) under the modified dynamics, we consider a stochastic process of the following general form:

\[ dv_t = \alpha_t(v_t) \, dt + \beta_t(v_t) \, dW_t^v, \quad (3.29) \]

where \( \alpha_t(v_t) \) and \( \beta_t(v_t) \) are deterministic functions of \( v_t \). Our aim is to find the conditions that are required to be satisfied by \( \alpha_t(v_t) \) and \( \beta_t(v_t) \) so that we can obtain a tractable Laplace transform of \( R_k \) conditional on the activity rate process filtration \( (\mathcal{G}_t)_{t \geq \tau \geq 0} \). A possible way is to express \( \int_{t_k}^{t_{k+1}} \sqrt{v_s} \, dW_s^v \) as a linear function of \( \int_{t_k}^{t_{k+1}} v_s \, ds \), \( v_{t_{k+1}} \) and \( v_{t_k} \), so that we can still perform analytical integration with respect to \( y \). First, we assume that after an appropriate transformation of variables, eq. (3.29) can be expressed as

\[ dP(v_t) = Q(v_t) \, dt + \sqrt{v_t} \, dW_t^v, \]

where \( P(v_t) \) and \( Q(v_t) \) are some functions of \( v_t \) to be determined. Considering \( P(v_t) \) as a function of the \( v_t \) process and applying Ito’s lemma, we can write the above SDE as

\[ dP(v_t) = [P'(v_t)\alpha_t(v_t) + \frac{1}{2}P''(v_t)\beta_t^2(v_t)] \, dt + \beta_t(v_t)P'(v_t) \, dW_t^v. \]

Matching the volatility terms of these two SDEs and solving for \( P(v_t) \) and \( Q(v_t) \), we obtain

\[ P(v_t) = \int_0^{v_t} \frac{\sqrt{z}}{\beta(z)} \, dz, \quad \text{and} \quad Q(v_t) = P'(v_t)\alpha_t(v_t) + \frac{1}{2}P''(v_t)\beta_t^2(v_t). \]

Therefore, the correlated diffusion term in eq. (3.28) can be expressed in terms of \( P(v_t) \) and \( Q(v_t) \) as follows

\[ \int_{t_k}^{t_{k+1}} \sqrt{v_s} \, dW_s^v = P(v_{t_{k+1}}) - P(v_{t_k}) - \int_{t_k}^{t_{k+1}} Q(v_s) \, ds. \]

To retain analytic tractability, it is necessary to choose \( Q(v_t) \) to be a linear function of \( v_t \) such that \( Q(v_t) = av_t + b_t \) for some constant \( a \) and time-dependent function \( b_t \) (see later discussion). The CIR process and 3/2-process are seen to satisfy this requirement. For the CIR defined in eq. (2.3), we can deduce that

\[ P(v_t) = \frac{v_t}{\epsilon} \quad \text{and} \quad Q(v_t) = \frac{k(\theta - v_t)}{\epsilon}. \quad (3.30) \]

For the 3/2-process defined in eq. (2.5), the corresponding \( P(v_t) \) and \( Q(v_t) \) are found to be

\[ P(v_t) = \frac{\ln v_t}{\sigma} \quad \text{and} \quad Q(v_t) = -\left(\frac{\tilde{q}}{\sigma} + \frac{\sigma}{2}\right)v_t + \frac{p}{\sigma}. \quad (3.31) \]

Substituting the above linear form of \( Q(v_t) \), we have

\[ \int_{t_k}^{t_{k+1}} \sqrt{v_s} \, dW_s^v = P(v_{t_{k+1}}) - P(v_{t_k}) - a \int_{t_k}^{t_{k+1}} v_s \, ds - \int_{t_k}^{t_{k+1}} b_s \, ds. \quad (3.32) \]

The conditional moment generating function of \( R_k \) is given by

\[
\mathbb{E}[e^{uR_k} | \mathcal{G}_T] = \mathbb{E} \left[ \exp \left( u((r-q)\Delta t_k + X_{T_k} - X_{T_{k-1}} + \rho \int_{t_{k-1}}^{t_k} \sqrt{v_s} \, dW_s^v) \right) \right] \\
= \exp \left( u \int_{t_{k-1}}^{t_k} c_s \, ds + \rho u[P(v_{t_k}) - P(v_{t_{k-1}})] + \xi(u) \int_{t_{k-1}}^{t_k} v_s \, ds \right),
\]
where \( c_t = r - q - \rho b_t \) and \( \xi(u) = -\psi_X(-iu) - apu \). Using the randomization method presented in Section 3, we can derive the conditional Laplace transform of \( R^2_k \) as

\[
\Phi_{k|G}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left( ix\sqrt{2\lambda} \int_{t_{k-1}}^{t_k} c_s \, ds + \xi(ix\sqrt{2\lambda}) \int_{t_{k-1}}^{t_k} v_s \, ds + i\rho x \sqrt{2\lambda} \left[ P(v_{t_k}) - P(v_{t_{k-1}}) \right] - x^2/2 \right) \, dx.
\]

By a similar argument, the new recursive formula for \( g_k(\lambda, v) \) is obtained as follows:

\[
g_k(\lambda, v) = \int_0^{\infty} \frac{g_{k+1}(\lambda, v_{t_k})}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(v_k, \Delta t_k; v, -\xi(ix\sqrt{2\lambda})) \exp \left( ix\sqrt{2\lambda} \int_{t_{k-1}}^{t_k} c_s \, ds + i\rho x \sqrt{2\lambda} \left[ P(v_{t_k}) - P(v_{t_{k-1}}) \right] - x^2/2 \right) \, dx \, dv_k. \tag{3.33}
\]

### 4 Numerical tests

In this section, we present the numerical tests for our proposed algorithms on the variance option prices and the fair strikes of volatility swaps. We use the NIG and VG models with the CIR process as the activity rate process as demonstrative examples. The Gaussian quadrature is used for the evaluation of the “transition matrix”. We compare the values obtained from our numerical algorithms to those obtained from the Monte Carlo simulation. We also investigate the effects of the correlation coefficient \( \rho \) and the volatility of the activity rate process \( \epsilon \) on the option prices.

The simulation of the sample paths for the general time-changed Lévy processes is not as simple as the Heston model, whereby the Euler scheme can be adopted directly. For general pure jumps processes, the jumps in the sample paths can be approximated by a number of Poisson processes. For some particular pure jump processes, we can apply time changes on a Brownian motion with drift by a random variable (Schoutens, 2003). For the NIG model, the large jumps are approximated by a number of independent Poisson processes representing different mean sizes of jumps and the small jumps by a Brownian motion. Asmussen and Rosinski (2001) prove a necessary condition for validity of the approximation of small jumps by a continuous Brownian motion.

However, such condition is not satisfied in the VG model, so a larger number of intervals are required for accurate approximation as the realized variance is sensitive to the percentage changes of the stock price. Therefore, the simulation time taken for the VG process is significantly longer than that of the NIG process. We followed the approach by Schoutens and Symens (2003) in the Monte Carlo simulation, where 100,000 sample paths for the NIG-CIR model and 10,000 paths for the VG-CIR model are simulated in our calculations. The numerical method was coded in Mathematica to take advantage of its built-in Bessel functions and simulations were run in Matlab. Both computer programs were executed with parallel algorithm in a multi-cores Intel i7 PC.

In our numerical calculations, we adopted the set of parameter values from Schoutens and Symens (2003). The values of the model are listed in Table 1, which are obtained via calibration to the mid-prices of a set of European call options on the S&P500 index on April 18, 2002 for the time-changed Lévy processes. We assume that \( q = 0, S_0 = 1 \) and \( r = 0.04 \) and take \( \Delta t = 1/252 \) for daily monitoring and \( \Delta t = 1/52 \) for weekly monitoring. We assume \( \rho = 0 \) when we compute the variance options prices first, and investigate the impact of correlation coefficient \( \rho \) and the volatility of the activity rate process \( \epsilon \) on the option prices afterwards. Our numerical experiences suggest that 36 grid points of \( x \) in eq. (3.22)
for the range of \([-7, 7]\) and 100 points for \(\lambda\) within \([0, 5000]\) in the Gaussian quadrature are sufficient for pricing daily monitoring variance options with \(N\) less than 40. Given the set of parameter values in Table 1, the range of \(v\) is chosen to be \([0, 5]\) with 36 grid points. Also, we take \(\lambda_r = 4\) in eq. (3.24) and (3.26) in the calculation of the inverse Laplace transform.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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</thead>
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<tr>
<td>(\alpha)</td>
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</tr>
<tr>
<td>(\beta)</td>
<td>-4.84</td>
</tr>
<tr>
<td>(\delta)</td>
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</tr>
<tr>
<td>(k)</td>
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</tr>
<tr>
<td>(\theta)</td>
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<tr>
<td>(\epsilon)</td>
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<td>(v_0)</td>
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</tbody>
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<table>
<thead>
<tr>
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<th>Value</th>
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<td>(G)</td>
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<tr>
<td>(M)</td>
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<tr>
<td>(k)</td>
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<td>(\theta)</td>
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<tr>
<td>(\epsilon)</td>
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<tr>
<td>(v_0)</td>
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</tbody>
</table>

Table 1: Values of the model parameters of the NIG-CIR and VG-CIR models.

**Assessment of numerical accuracy**

Tables 2 and 3 show the discrete variance put option prices with daily sampling frequency. Our numerical method is quite effective with an average computational time of 110 seconds. The Monte Carlo simulation times vary with the number of monitoring dates and can go up to hours for longer maturity dates. Note that the computational times for different \(N\) in our algorithms are similar since the time required for the computation of the \(N\)-power of a 36x36 matrix \((h_{ij})\) is very small compared to that of computing all the entries of the matrix. With \(N\) less than 40, the numerical errors are less than 1% across different strikes and monitoring dates. Both models obtain similar prices for options with the same strike.

<table>
<thead>
<tr>
<th>NIG-CIR model</th>
<th>Strike (ATM)</th>
<th>Numerical Scheme</th>
<th>Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>N (days)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.027807</td>
<td>0.018819</td>
<td>117.34</td>
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<td>15</td>
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<td>20</td>
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<td>40</td>
<td>0.028070</td>
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<th>VG-CIR model</th>
<th>Strike (ATM)</th>
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<td>30</td>
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</tr>
<tr>
<td>40</td>
<td>0.028239</td>
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<td>113.97</td>
</tr>
</tbody>
</table>

Table 2: Comparison of at-the-money (ATM) discrete variance put options prices obtained from the numerical algorithm and Monte Carlo simulation under daily monitoring. The computational times are measured in units of second. In the Monte Carlo simulation, we used 100,000 paths for the NIG-CIR model, and 10,000 paths for the VG-CIR model. Here, SE stands for the standard error in the simulation.
Table 3: Comparison of out-of-the-money (OTM) and in-the-money (ITM) discrete variance put options prices obtained from the numerical algorithm and Monte Carlo simulation under daily monitoring. In the Monte Carlo simulation, we used 100,000 paths for the NIG-CIR model, and 10,000 paths for the VG-CIR model. Here, SE stands for the standard error in the simulation. The OTM strikes are taken to be 80% of the ATM strikes, and 120% for the ITM strikes.

For weekly sampling variance put options, a wider range of \( \nu \) is required and the computational times are longer than under daily monitoring. Table 4 lists the option prices under the NIG-CIR model. The pricing errors are mostly less than 1% for small \( N \) and can go up to a few percents for larger \( N \). The errors can be reduced by taking more grid points, but the required CPU running times may increase quite substantially.

Table 5 presents the fair strikes of the discrete volatility swaps in the two models. The pricing errors are still less than 1%. We also note that the fair strikes of the volatility swaps computed from the NIG-CIR model are always larger than those obtained from the VG-CIR model for this set of parameters. The strike prices are increasing with the number of monitoring dates. This may be due to the sample paths will encounter big jumps with a larger probability in a longer time period, and contributing more to the realized variance. However, large jumps may set the put options into out-of-the-money.

Though our numerical algorithms work well with good accuracy and efficiency for short-maturity derivatives, we remark that numerical accuracy deteriorates as the sampling frequency becomes significantly large. One has to take denser grid points for the discretization and pay for more computational costs since the realized variance is much more sensitive to the percentage changes of stock price than stock price itself due to the squared terms in the realized variance. Under this scenarios, it may be advisable to use the quadratic variation approximation for pricing discrete variance derivatives with long-maturity and high sampling frequency.
<table>
<thead>
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<th>NIG-CIR model</th>
<th>N (weeks)</th>
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<th>8</th>
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<td>Strike (ITM)</td>
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<tr>
<td>Monte Carlo</td>
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<td>0.000038</td>
<td>0.000037</td>
<td>0.000036</td>
<td>0.000034</td>
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</tr>
<tr>
<td>SE</td>
<td>0.019583</td>
<td>0.017147</td>
<td>0.016255</td>
<td>0.015366</td>
<td>0.014758</td>
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</tr>
<tr>
<td>Strike (OTM)</td>
<td>0.019513</td>
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<td>0.016312</td>
<td>0.015688</td>
<td>0.015324</td>
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<td>Numerical Scheme</td>
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<tr>
<td>SE</td>
<td>0.019583</td>
<td>0.017147</td>
<td>0.016255</td>
<td>0.015366</td>
<td>0.014758</td>
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</tr>
</tbody>
</table>

Table 4: Comparison of the numerical values of the discrete variance put options prices obtained from our numerical algorithm and Monte Carlo simulation under weekly monitoring. In the Monte Carlo simulation, we used 100,000 paths for the NIG-CIR model, and 10,000 paths for the VG-CIR model. Here, SE stands for the standard error in the simulation. The OTM (ITM) strikes are taken to be 80% (120%) of the ATM strikes.

<table>
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<tr>
<th>NIG-CIR model</th>
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<th>5</th>
<th>10</th>
<th>15</th>
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<table>
<thead>
<tr>
<th>VG-CIR model</th>
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</thead>
<tbody>
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<td>Numerical Scheme</td>
</tr>
<tr>
<td>MC</td>
</tr>
<tr>
<td>SE</td>
</tr>
</tbody>
</table>

Table 5: Comparison of the numerical values of the fair strike prices of the discrete volatility swaps obtained from our numerical algorithm and Monte Carlo simulation under daily monitoring. In the Monte Carlo simulation, we used 100,000 paths for the NIG-CIR model, and 10,000 paths for the VG-CIR model. Here, SE stands for the standard error in the simulation.
To examine the pricing behavior of the variance put option and volatility swap under the NIG model, we plot the prices of the variance put option against $\alpha$ in Figure 1 and the fair strike values of the volatility swap against $\alpha$ in Figure 2, where both derivative products have maturity of 10 days and daily monitoring frequency. For the variance put option, the at-the-money strike is taken to be 0.028018. The variance put option price is seen to be increasing with respect to $\alpha$ while the fair strike of the volatility swap decreases with $\alpha$. To explain the observed pricing properties, it is seen that the distribution of the NIG process becomes less positive-skewed when $\alpha$ increases. The mean of the distribution decreases and more weight of the distribution is shifted to the left hand side. This leads to an increasing trend of the variance option price with respect to $\alpha$. Also, the fluctuation of the stock price process decreases with increasing value $\alpha$, so the fair strike of the volatility swap decreases as $\alpha$ increases.

We performed similar analysis of pricing behavior of the same pair of derivative products under the VG-CIR model. We plot the prices of the variance put option prices against $C$ in Figure 3 and the fair strike values of the volatility swap against $C$ in Figure 4. For the variance put option, the at-the-money strike is now taken to be 0.027502. Unlike the NIG process, the mean and variance of the VG-CIR distribution both increase with $C$. This explains why the variance put option price decreases with increasing value $C$ while the fair strike of the volatility swap increases when $C$ increases.

The correlation of the stock price process and the activity rate process can be introduced only if both of the processes contain diffusion components or jump components at the same time. Here, we consider the Heston model to investigate the effect of the correlation structure in the stock price dynamics [see eq. (3.28)]. The parameters of the Heston model are taken from Zhu and Lian (2011) and they are listed as follows: $\theta = 0.022$, $k = 11.35$, $v_0 = 0.04$ and $\epsilon = 0.618$. Figure 5 shows the discrete variance option prices against $\rho$ while Figure 6 shows the fair strikes of volatility swap, both with three different values of $\epsilon$. We observe that the put option price increases with $\epsilon$ while the fair strike of volatility swap decreases with $\epsilon$. This may be due to the phenomenon that the discrete realized variance is roughly constant with $\epsilon$ (Zheng and Kwok, 2014c) while the term $\text{var}(V_d)$ in eq. (3.27) increases with $\epsilon$, so the fair strike of the volatility swap decreases.

5 Conclusion

We have proposed numerical algorithms for pricing variance options and volatility derivatives with short-maturity and/or low sampling frequency under the time-changed Lévy processes. The time-changed Lévy processes include a broad range of other models as subclass. They can capture stochastic volatility given their wider jumps structure and they also provide reasonable analytic tractability for pricing discrete variance derivatives. By employing the randomization formula, the computation of the moment generating function of the realized variance simplifies to the calculation of a “transition matrix” in which each entry is the expectation of the transition probability of different levels of volatility. With the analytical formula of the “transition matrix”, the computational dimension is reduced and the numerical calculation of the variance derivatives prices is made feasible. The algorithms compute the moment generating function across each of the monitoring dates by calculating the multiplication of the “transition matrix”. The variance derivatives prices are obtained given the moment generating function values at the grid points by inverse Laplace transform.

We performed numerical calculations of the NIG-CIR, the VG-CIR and the Heston models for pricing the variance options and volatility swaps. Our proposed numerical algorithms work particularly well for short-maturity variance derivatives with less computational cost.
than the Monte Carlo simulation. Across different strikes and monitoring dates in the discrete variance put options, most of the price errors are within 1%. Similar level of numerical accuracy for pricing volatility swaps are exhibited in our numerical tests. The numerical tests also demonstrate that the impacts of $\alpha$ of the NIG model and $C$ of the VG model on the option prices and the fair strikes of the volatility swaps. Besides, the impact of introducing the leverage effect in the Heston model is also presented with different levels of volatility of volatility.
REFERENCES


Appendix - Proof of Lemma 1

For the CIR process defined in eq. (2.3), one can derive the joint Laplace transform of \( v_t \) and \( \int_0^t v_s \, ds \) by following the standard procedure of solving the characteristic function in the affine models. Specifically, we let

\[
\phi(\lambda, \eta) = \mathbb{E} \left[ \exp \left( -\lambda v_\tau - \eta \int_0^\tau v_s \, ds \right) \bigg| v_0 \right] = \exp(A(\tau) + v_0 B(\tau)),
\]

one can obtain the Riccati equations for \( A(\tau) \) and \( B(\tau) \) (Jeanblanc et al., 2009). The functions \( A(\tau) \) and \( B(\tau) \) are given by

\[
A(\tau) = \frac{2k\theta}{\epsilon^2} \ln \left( \frac{2\delta e^{(\delta+k)\tau/2}}{e^2\lambda(e^{\delta\tau} - 1) + \delta(e^{\delta\tau} + 1) + k(e^{\delta\tau} - 1)} \right)
\]

\[
B(\tau) = -\frac{\lambda(\delta + k + e^{\delta\tau}(\delta - k)) + 2\eta(e^{\delta\tau} - 1)}{e^2\lambda(e^{\delta\tau} - 1) + \delta(e^{\delta\tau} + 1) + k(e^{\delta\tau} - 1)},
\]

where \( \delta = \sqrt{k^2 + 2\epsilon^2\eta} \). To simplify the notations, we define

\[
A_1 = 2\delta e^{(\delta+k)\tau/2}, \quad B_1 = \frac{1}{e^2(e^{\delta\tau} - 1)}, \quad C_1 = \delta(e^{\delta\tau} + 1) + k(e^{\delta\tau} - 1),
\]

\[
D_1 = -(\delta + k + e^{\delta\tau}(\delta - k))v_0, \quad E_1 = -2\eta(e^{\delta\tau} - 1)v_0 \quad \text{and} \quad \beta = \frac{2k\theta}{\epsilon^2}.
\]

Suppressing the dependence on \( \eta \), the joint characteristic function \( \phi(\lambda) \) can be written as

\[
\phi(\lambda) = \exp(B_1D_1) \left( \frac{A_1B_1}{\lambda + C_1B_1} \right)^\beta \exp \left( \frac{B_1(E_1 - B_1D_1C_1)}{\lambda + C_1B_1} \right).
\]

On the other hand, let \( p_{\text{CIR}}(v, y; \tau; v_0) \) be the joint transition density of \( v_\tau \) and \( \int_0^\tau v_s \, ds \) from \( (v_0, 0) \) at time 0 to \( (v, y) \) at time \( \tau \), then the characteristic function can also be expressed as

\[
\mathbb{E} \left[ \exp \left( -\lambda v_\tau - \eta \int_0^\tau v_s \, ds \right) \bigg| v_0 \right] = \int_0^\infty \int_0^\infty e^{-\lambda v - \eta y} p_{\text{CIR}}(v, y, \tau; v_0) \, dy \, dv
\]

\[
= \int_0^\infty e^{-\lambda v} G_{\text{CIR}}(v, \tau; v_0, \eta) \, dv.
\]

We may view \( \exp((A(\tau) + v_0 B(\tau)) \) as the Laplace transform of \( G_{\text{CIR}}(v, \tau; v_0, \eta) \) on \( v \). To recover the value of \( G_{\text{CIR}} \), we apply the inverse Laplace transform formula

\[
\mathcal{L}^{-1} \left[ p^{-\nu-1}e^{a/p} \right](x) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{px} \left( p^{-\nu-1}e^{a/p} \right) \, dp
\]

\[
= \left( \frac{x}{a} \right)^{\nu/2} I_\nu(2\sqrt{ax}),
\]

where \( \nu > -1 \), \( a \) is a constant and \( I_\nu \) is the modified Bessel function of the first kind of order \( \nu \). Substituting \( \phi(\lambda) \) into the above formula, we obtain
\[ G_{\text{CR}}(v, \tau; v_0, \eta) \]
\[ = \mathcal{L}^{-1}[\phi(\lambda)](v) \]
\[ = \mathcal{L}^{-1}\left[ \exp(B_1 D_1) \left( \frac{A_1 B_1}{\lambda + C_1 B_1} \right)^{\beta} \exp \left( \frac{B_1(E_1 - B_1 D_1 C_1)}{\lambda} \right) \right](v) \]
\[ = (A_1 B_1)^{\beta} e^{(B_1 D_1 - C_1 B_1 v)} \mathcal{L}^{-1}\left[ \left( \frac{1}{\lambda} \right)^{\beta} \exp \left( \frac{B_1(E_1 - B_1 D_1 C_1)}{\lambda} \right) \right](v) \]
\[ = (A_1 B_1)^{\beta} e^{(B_1 D_1 - C_1 B_1 v)} \left[ \frac{v}{B_1(E_1 - B_1 D_1 C_1)} \right]^{(\beta-1)/2} I_{\beta-1} \left( 2 \sqrt{B_1(E_1 - B_1 D_1 C_1)v} \right) \]
\[ = 2 \exp \left( e^2 k(v_0 - v) - \frac{(1 + e^{\delta \tau})(v + v_0) \delta^2}{e^{\delta \tau} - 1} \right) \left[ \frac{e^{-\delta \tau}(e^{\delta \tau} - 1)^2 v^4}{v_0 \delta^2} \right]^{\frac{\beta-1}{2}} \]
\[ \left[ \frac{e^{\frac{1}{2}(k + \delta \tau) \delta}}{(e^{\delta \tau} - 1) e^2} \right]^{\beta} I_{\beta-1} \left( 4 \sqrt{\frac{e^{\delta \tau} v_0 \delta^2}{(e^{\delta \tau} - 1)^2 e^4}} \right). \]
Figure 1: Plot of the prices of the 10-day variance put option with daily monitoring against $\alpha$ of the NIG-CIR model. The at-the-money strike is taken to be 0.028018.

Figure 2: Plot of the fair strikes of the 10-day volatility swap with daily monitoring against $\alpha$ of the NIG-CIR model.

Figure 3: Plot of the prices of the 10-day variance put option with daily monitoring against $C$ of the VG-CIR model. The at-the-money strike is taken to be 0.027502.

Figure 4: Plot of the fair strikes of the 10-day volatility swap with daily monitoring against $C$ of the VG-CIR model.
Figure 5: Plot of the prices of the 10-day variance put options with daily monitoring against the correlation coefficient $\rho$ with three different values of $\epsilon$ in the Heston model. The at-the-money strike is taken to be 0.03640.

Figure 6: Plot of the fair strikes of the 10-day volatility swaps with daily monitoring against the correlation coefficient $\rho$ with three different values of $\epsilon$ in the Heston model.