

Numerical pricing of CoCo bonds with Parisian trigger feature using the Fortet method

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Abstract

Unlike conventional convertible bonds, contingent convertible (CoCo) bonds are converted into equity shares of the issuing bank subject to certain trigger mechanisms (accounting and/or regulatory trigger) when the issuing bank is under financial nonviable state. We consider pricing of these CoCo bonds using the contingent claims approach, where the state variables are the stock price and Tier 1 capital ratio. We use the Parisian feature to model the regulatory trigger where equity conversion is triggered when the capital ratio stays under the nonviable state consecutively for a certain period of time. The accounting trigger is modeled using the one-touch barrier feature associated with the capital ratio. The Parisian trigger feature adds one extra path dependent state variable in the pricing model of a CoCo bond. We design effective numerical algorithms for pricing the CoCo bonds using the extended Fortet method that avoid adding one extra state variable for the Parisian feature of regulatory trigger. Pricing properties of the CoCo bonds under both regulatory trigger and accounting trigger are explored.

Keywords: CoCo bonds, conversion triggers, Parisian feature, Fortet method

1 Introduction

The contingent convertible (CoCo) bonds are hybrid equity-credit securities that are designed to have the loss absorption mechanism, where the CoCo bond is converted into shares of the issuing bank when the bank is under financial nonviable state. This equity conversion feature serves to lower the debt/equity ratio when the bank is in the state of nonviability. Another loss absorption mechanism is via predefined principal writedown of the principal. Since their first launch in 2009, the growth of the issuance of CoCo bonds has been quite phenomenal, with issuance amount more than 120 billion US dollars by 2015. From the investors' perspective, CoCo bonds pay a high coupon rate that compensates the potential loss absorption liabilities. From the issuer's perspective, the CoCo bonds can be counted as

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additional Tier 1 capital even under the more stringent regulatory requirement imposed after the financial tsunami in 2008. Regulators have been demanding more loss absorbency within hybrid capital, holding the view that investors of these capital instruments can contribute to the recovery phase of a distressed firm.

The loss absorption mechanism is activated either by regulator's discretion based on judgment about solvency prospects of the issuing bank and / or mechanical trigger based on the capital ratio that falls below some preset threshold value. The capital ratio trigger can be either set contractually in terms of the book value of the Common Equity Tier 1 capital as a ratio of the risk-weighted assets or the market value of the ratio of the bank's stock market capitalization to its assets. The details of these trigger mechanisms can be found in Avdjiev *et al.* (2013). There are subtleties in the actual implementation of these trigger mechanisms. Due to the general lack of transparency on the criteria used by regulators in activating the discretionary trigger, timing of the activation is highly uncertain. There may be a lagged effect in the activation of the book value trigger since accounting reports are published infrequently. Glasserman and Nouri (2012) present a thorough discussion on the risk implication of CoCo bonds under various conversion mechanisms.

The pricing of CoCo bonds poses great challenge, in particular, the modeling of uncertainty of the timing of conversion trigger. Wilkens and Bethke (2014) present an empirical assessment of selected pricing models of CoCo bonds. They categorize the pricing models into three types: structural models, equity derivatives models and credit derivatives models. The structural models use the fundamental information of issuing banks asset and liabilities structure, and impose contingent capital conversion into equity when certain trigger mechanism is activated. However, equity value is an option on the firm value, so optionality on equity value becomes a compound option on the firm value process. Brigo *et al.* (2015) use the structural-default approach that considers two different thresholds for the firm value, the first one triggering equity conversion and the second one triggering default. On the other hand, Wilkens and Bethke (2014) advocate the use of equity derivatives models due to its straightforward modeling and interpretation of the conversion mechanism. For example, De Spiegeleer and Schoutens (2012) model conversion trigger by the first time that the stock price crosses a barrier level from above in their equity derivatives model. In general, equity derivatives models have better tractability due to its simplicity as one-dimensional pricing models since the stock price is the only single state variable. However, this may not reflect the contractual reality in CoCo bonds where conversion into equity is triggered based on the capital ratio falling below a preset threshold. In the credit derivatives models, conversion trigger is based on the credit spreads or the risk premia of credit default swaps (CDS). Pelger (2012) argue that defining the conversion trigger in terms of credit spreads or CDS premia may circumvent unobservability of the firm value process. With the advantage of rapid adjustment to new information, credit spreads and CDS premia reflect new changes in the firm value in a timely fashion. Using the reduced form approach, Cheridito and Xu (2015a) model the arrival of the conversion trigger in a CoCo bond by the first jump of a time-changed Poisson process.

In this paper, we propose a two-state pricing model using the joint stochastic process of the Tier 1 capital ratio and stock price as the underlying state variables. The capital ratio is included in the pricing model since it appears as part of the contractual specification.

Under Basel III, the minimum trigger level in terms of the ratio of Common Equity Tier 1 capital to risk-weighted assets required for a CoCo bond to be qualified as Additional Tier 1 capital is 5.125%. Many CoCo bonds have set their trigger levels at this value. Since equity conversion occurs upon activation of trigger, so it is more direct to use the stock price as the underlying state variable rather than the firm value process. We would like to incorporate both the mechanical trigger and regulatory trigger in our CoCo bond pricing model. Mechanical trigger is activated when the capital ratio hits the one-touch threshold level from above. The regulatory trigger is modeled by the event of breaching the allowable cumulative occupation time that the capital ratio is staying below some threshold value that warns against potential nonviability of the issuing bank. The use of consecutive occupation time in modeling regulatory trigger follows a similar approach used in modeling default risk in defaultable debts using the notion of occupation times (Makarov *et al.*, 2015). This allows for a situation in which the issuing firm may be temporarily in the state of financial distress. The regulatory trigger occurs only when the excursion time period of staying in distressed state exceeds certain level. In our CoCo bond pricing model, we set the warning threshold to be above the mechanical trigger threshold of mandatory equity conversion. The regulatory trigger mechanism resembles the cumulative Parisian feature on the knock-out condition in barrier style derivatives. The dimensionality of the pricing model increases by one due to an inclusion of this extra Parisian state variable.

Following the approach in Cheridito and Xu (2015b) and Chung and Kwok (2016), we model the capital ratio directly as a diffusion process and consider the random times of breaching the various thresholds as first passage times under the joint equity-capital ratio process. Unlike the stock price process which can be observed continuously, the capital ratio process is observed only infrequently. This may pose technical challenges in performing calibration of the parameters. Ritzema (2015) suggests the use of CDS rates in the calibration procedure. Cheridito and Xu (2015b) propose an improved procedure of computing the Q-survival probabilities inferred from the CDS rates using a reduced form method. The papers by Ritzema (2015) and Cheridito and Xu (2015b) provide details on the procedures of calibrating the capital ratio. They also consider pricing and hedging CoCo bonds using tradeable securities, like stocks, credit default swaps and interest rate swaps, to hedge against various risk factors in the joint equity-capital ratio models. De Spiegeleer *et al.* (2017) propose the use of observable CoCo bond prices to calibrate the implied capital ratio volatility. Since calibration of the model parameters can be performed using tradeable securities, this justifies the use of a risk neutral measure in the pricing models.

In the design of an effective numerical algorithm for computing the fair value of a CoCo bond, we adopt the Fortet method that was first used for pricing defaultable debts under stochastic interest rates (Longstaff and Schwartz, 1995; Collin-Dufresne and Goldstein, 2001). Later, the Fortet method was extended by Bernard *et al.* (2005, 2006, 2008) in their series of papers on pricing various types of barrier style derivatives under stochastic interest rates. Another application of the Fortet method is presented by Coculescu *et al.* (2008) in pricing defaultable sensitive claims under imperfect information. The Fortet method solves an integral equation for the determination of the first passage time to a barrier under the joint Gaussian process by making use of the strong Markov property of the underlying Gaussian process. We manage to extend the Fortet method in pricing a CoCo bond in two aspects.

Firstly, we deal with two barriers, the upper warning threshold and the lower mandatory trigger threshold on the capital ratio. Secondly, the curse of an additional Parisian state variable in the CoCo bond pricing model is resolved. The computational complexity of our extended Fortet method remains to be the same as that of the pricing algorithm for pricing one-touch single-barrier options under joint Gaussian process.

The paper is organized as follows. In the next section, we use the equity derivatives approach to construct the pricing model of a CoCo bond under both mechanical and regulatory trigger mechanisms. The stochastic process of the Tier 1 capital ratio is modeled as a mean reversion Geometric Brownian motion that is correlated with the Geometric Brownian motion of the stock price process. The pricing model includes the usual cash flows of coupons and terminal par payment, together with the feature of conversion into equity under the two conversion mechanisms. The equity conversion is activated either under the one-touch barrier criterion (mechanical trigger) or the Parisian cumulative occupation time criterion (regulatory trigger) based on the path realization of the stochastic process of the Tier 1 capital ratio. In Section 3, we consider an extension of the Fortet method to devise an effective numerical algorithm for pricing a CoCo bond with the Parisian feature and one-touch barrier feature of equity conversion. We achieve dimension reduction in the pricing algorithm via the determination of the auxiliary CoCo bond value function at the Tier 1 capital ratio that is slightly below the warning threshold of non-viability. In Section 4, we present numerical pricing results and performance analysis of the algorithm. We also present numerical studies of the pricing properties of the CoCo bonds under various market conditions. Summary and conclusive remarks are presented in the last section.

2 Model formulation

Similar to the pricing model of conventional convertible bonds, the value of a CoCo bond is decomposed into the bond component and equity component. Unlike conventional convertible bonds where conversion into equity occurs when the underlying stock price is performing well, conversion into shares of the issuing bank in a CoCo bond occurs when the Tier 1 capital ratio either stays below the warning threshold G consecutively for a period of d (years) or hits an one-touch conversion barrier level B , where $B < G$. These two trigger mechanisms of mandatory equity conversion are called the Parisian trigger and one-touch trigger, respectively. Provided that no trigger event is activated throughout the contractual life of the CoCo bond, the CoCo investor is entitled to receive the bond par F at bond's maturity and the coupon payment stream at the annualized rate of c for the whole life of the bond. When the CoCo bond is terminated prematurely due to either one of the two conversion triggers, the bond is converted into N shares of the underlying stock. It is common to set a floored value K on the stock price at equity conversion so that the CoCo bond value upon conversion has a lower floor value. The payoff of equity conversion at the random conversion time τ is given by $N \max(S_\tau, K)$.

Let S_t and Y_t denote the price process of the underlying stock and the logarithm of the value process of the Tier 1 capital ratio, respectively. Under a risk neutral measure Q , the

dynamics of S_t and Y_t are assumed to evolve as follows:

$$\frac{dS_t}{S_t} = (r - q) dt + \sigma \left(\sqrt{1 - \rho^2} dW_t^1 + \rho dW_t^2 \right), \quad (2.1a)$$

$$dY_t = \alpha(\mu - Y_t) dt + \sigma_Y dW_t^2, \quad (2.1b)$$

where S_t follows the usual Geometric Brownian motion and Y_t follows the mean reversion process with mean reversion level μ and reversion rate α , W_t^1 and W_t^2 are two independent standard Brownian motions. Here, the correlation coefficient ρ , riskfree interest rate r , dividend yield q , stock price volatility σ , Tier 1 capital volatility σ_Y , μ and α are assumed to be constant.

We let T be the maturity date of the CoCo bond and time zero be the time of initiation of the CoCo bond. At the time of issuance of the CoCo bond, the issuing bank should not be under the nonviable state; so we observe $Y_0 > \ln G > \ln B$. One-touch trigger occurs at the first random time τ_B at which Y_t hits the logarithm of the lower barrier $\ln B$ from above; that is,

$$\tau_B = \inf \{s \geq 0 : Y_s = \ln B | S_0, Y_0\}. \quad (2.2a)$$

On the other hand, monitoring of Y_t staying below the warning threshold $\ln G$ is initiated at the first passage time that Y_t hits $\ln G$ from above; and monitoring of nonviability is discontinued once Y_t goes above $\ln G$. Regulatory equity conversion (Parisian trigger) occurs when Y_t stays below $\ln G$ consecutively for a period of d years, conditional on Y_t staying above $\ln B$ throughout. Let $H_{t, \ln G}$ denote the excursion time that the log capital ratio process Y_t stays below $\ln G$ in its most recent excursion to the region below $\ln G$; that is,

$$H_{t, \ln G} = (t - h_{t, \ln G}) \mathbf{1}_{\{Y_t \leq \ln G\}},$$

where

$$h_{t, \ln G} = \sup \{u \leq t : Y_u = \ln G\}.$$

Given the time period d , the right-continuous inverse of the excursion time is defined by

$$\tau_P = \inf \{t \geq 0 : H_{t, \ln G} \geq d\}. \quad (2.2b)$$

That is, τ_P is the first random time at which the occupation time to the region below $\ln G$ reaches d years so that the Parisian trigger occurs. As a summary, the CoCo bond terminates due to one-touch trigger when $\tau_B < \min\{\tau_P, T\}$ and regulatory equity conversion occurs provided that $\tau_P < \min\{\tau_B, T\}$. The CoCo bond survives until maturity T , provided that $T < \min\{\tau_P, \tau_B\}$.

We let c_i be the discrete coupon amount received on the coupon payment date t_i , $i = 1, 2, \dots, n$, where $c_i = c(t_i - t_{i-1})F$. Here, c denotes the constant annualized coupon rate of the CoCo bond and F is the bond par. We take $t_0 = 0$ and $t_n = T$ for convenience. The discrete coupon c_i is received at t_i , $i = 1, 2, \dots, n$, and the par F is received at T , provided that the CoCo bond survives at t_i and T , respectively. The sum of the coupons and par in present value received contingent upon survival of the CoCo bond gives the bond component of the CoCo bond. The equity component of the bond is given by the present value of the shares received upon mandatory conversion under either the Parisian trigger or one-touch

trigger. Under the risk neutral pricing measure Q , the time-0 value of the CoCo bond is given by

$$\begin{aligned} V_0(S_0, Y_0; T) = \mathbb{E}_Q & \left[\sum_{i=1}^n c_i e^{-rt_i} \mathbf{1}_{\{t_i < \min\{\tau_P, \tau_B\}\}} + e^{-rT} F \mathbf{1}_{\{T < \min\{\tau_P, \tau_B\}\}} \right. \\ & + N \max\{S_{\tau_P}, K\} e^{-r\tau_P} \mathbf{1}_{\{\tau_P \leq \min\{\tau_B, T\}\}} \\ & \left. + N \max\{S_{\tau_B}, K\} e^{-r\tau_B} \mathbf{1}_{\{\tau_B \leq \min\{\tau_P, T\}\}} | S_0, Y_0 \right], \end{aligned} \quad (2.3)$$

where $Y_0 > \ln G > \ln B$ and K is the floored value of the stock price at equity conversion. In our subsequent exposition, it is convenient to use $X_t = \ln S_t$ as the state variable, where the dynamics of X_t is governed by

$$dX_t = \left(r - q - \frac{\sigma^2}{2} \right) dt + \sigma \left(\sqrt{1 - \rho^2} dW_t^1 + \rho dW_t^2 \right),$$

so that X_t and Y_t is a joint Gaussian process.

We denote the time- t value function of the CoCo bond by $V_t(X_t, Y_t; T)$. The determination of the joint density of (τ_P, X_{τ_P}) poses the major challenge in the pricing calculation of $V_0(X_0, Y_0; T)$. We would like to explore an alternative pricing approach to resolve the technical difficulties. The issuing firm enters into the monitoring state of financial nonviability when Y_t falls a small distance ε below the warning threshold $\ln G$ from above. We define the random first passage time of Y_s , $s \geq t$, hitting $\ln G - \varepsilon$ from above by

$$\tau_{G,\varepsilon}^- = \inf\{s \geq t : Y_s = \ln G - \varepsilon | X_t, Y_t\}, \quad Y_t > \ln G. \quad (2.4a)$$

Under the continuous time model, we take the limit $\varepsilon \rightarrow 0^+$, and define

$$\tau_G^- = \lim_{\varepsilon \rightarrow 0^+} \tau_{G,\varepsilon}^-. \quad (2.4b)$$

We introduce $\tau_{G,\varepsilon}^-$ as an immediate step since we adopt a finite but small ε in our later numerical pricing algorithms. We adopt the notation: $(\ln G)^\pm = \lim_{\varepsilon \rightarrow 0^+} \ln G \pm \varepsilon$. Since the Parisian feature based on the consecutive occupation time staying below $\ln G$ is activated once the logarithm of capital ratio Y_t crosses $\ln G$ from above, it is instructive to define the auxiliary value function of the CoCo bond to be $V_t^G(X_t; T)$ as follows:

$$V_t^G(X_t; T) = V_t(X_t, (\ln G)^-; T). \quad (2.5)$$

The key ingredient in our continuous time pricing formulation is to express $V_t(X_t, Y_t; T)$ in terms of $V_t^G(X_t; T)$, coupons and par contingent on survival. The pricing calculations mainly involve the joint density of $(\tau_G^-, X_{\tau_G^-})$, the numerical approximation of which can be obtained by the Fortet method. Assuming $Y_t > \ln G$, the time- t value of the discrete coupons received within the time period $(t, s]$ is represented by

$$c(t, s) = \sum_{i=1}^n c_i e^{-r(t_i - t)} \mathbf{1}_{\{t_i \in (t, s]\}}.$$

Under the risk neutral measure Q , we let $f_G^-(u, x)$ denote the conditional joint density function of $(\tau_G^-, X_{\tau_G^-})$, where

$$f_G^-(u, x)dudx = Q(\tau_G^- \in (u, u + du), X_{\tau_G^-} \in (x, x + dx)|X_t, Y_t).$$

Conditional on $Y_t > \ln G$, based on the strong Markov property of the joint Gaussian process of X_t and Y_t , we deduce that

$$\begin{aligned} V_t(X_t, Y_t; T) &= \mathbb{E}_Q^t \left[[c(t, T) + e^{-r(T-t)}F] \mathbf{1}_{\{\tau_G^- > T\}} \right. \\ &\quad \left. + [c(t, \tau_G^-) + e^{-r(\tau_G^- - t)}V_{\tau_G^-}^G(X_{\tau_G^-}; T)] \mathbf{1}_{\{\tau_G^- \leq T\}} \right] \\ &= \int_{-\infty}^{\infty} [c(t, T) + e^{-r(T-t)}F] Q(\tau_G^- > T|X_t, Y_t) dx \\ &\quad + \int_t^T \int_{-\infty}^{\infty} [c(t, u) + e^{-r(u-t)}V_u^G(x; T)] f_G^-(u, x) dx du, \quad 0 \leq t \leq T, \end{aligned} \quad (2.6)$$

where \mathbb{E}_Q^t denotes the expectation under Q conditional on X_t and Y_t .

Next, we present the formulation of $V_t^G(X_t; T)$. There are 3 possible scenarios to be considered: (i) leaving the monitoring state at a later time before maturity, (ii) mechanical conversion into equity under the one-touch trigger, (iii) regulatory conversion into equity under the Parisian trigger. The quantitative modeling of these 3 scenarios is summarized below.

1. When the logarithm of capital ratio crosses $\ln G + \varepsilon$ from below prior to the occurrence of the Parisian trigger or one-touch trigger, the CoCo bond leaves the monitoring state and the clock of counting the excursion time of staying between $\ln G$ and $\ln B$ is set to be zero. For a finite but small value of ε , we define

$$\tau_{G,\varepsilon}^+ = \inf\{s \geq t; Y_s = \ln G + \varepsilon | X_t, Y_t = \ln G - \varepsilon\},$$

and the corresponding continuous time limit is defined by

$$\tau_G^+ = \lim_{\varepsilon \rightarrow 0^+} \tau_{G,\varepsilon}^+.$$

This scenario occurs when $\tau_G^+ < \min\{\tau_B, t + d\}$ and the value function at τ_G^+ becomes $V_{\tau_G^+}^+(X_{\tau_G^+}, (\ln G)^+; T)$.

2. Mechanical conversion into equity occurs at τ_B , where $\tau_B < \min\{\tau_G^+, t + d\}$. The corresponding payoff of the CoCo bond at τ_B is $N \max\{e^{X_{\tau_B}}, K\}$.
3. When the Tier 1 capital ratio stays between G and B consecutively for a time period of d , where $t + d < \min(\tau_B, \tau_G^+)$, regulatory conversion into equity occurs. The corresponding payoff of the CoCo bond at $t + d$ is $N \max\{e^{X_{t+d}}, K\}$.

We define the marginal density function of X_{t+d} conditional on $t + d < \min\{\tau_G^+, \tau_B\}$ to be

$$F_{G^+,B}(x)dx = Q(t + d < \min\{\tau_G^+, \tau_B\}, X_{t+d} \in (x, x + dx) | X_t, Y_t = \ln G^-).$$

The conditional joint density functions $f_{G^+}(u, x)$ and $f_B(u, x)$ are similarly defined by

$$f_{G^+}(u, x)dxdu = Q(\tau_G^+ \in (u, u + du), X_{\tau_G^+} \in (x, x + dx) | X_t, Y_t = \ln G^-) \mathbf{1}_{\{\tau_G^+ < \min\{\tau_B, t+d\}\}},$$

$$f_B(u, x)dxdu = Q(\tau_B \in (u, u + du), X_{\tau_B} \in (x, x + dx) | X_t, Y_t = \ln G^-) \mathbf{1}_{\{\tau_B < \min\{\tau_G^+, t+d\}\}}.$$

For $t + d < T$, the time- t auxiliary value function $V_t^G(X_t; T)$ is then given by

$$\begin{aligned} V_t^G(X_t; T) &= \int_t^{t+d} \int_{-\infty}^{\infty} f_{G^+}(u, x) [c(t, u) + e^{-r(u-t)} V_u(x, \ln G^+; T)] dxdu \\ &\quad + \int_t^{t+d} \int_{-\infty}^{\infty} f_B(u, x) [c(t, u) + e^{-r(u-t)} N \max\{e^x, K\}] dxdu \\ &\quad + \int_{-\infty}^{\infty} F_{G^+,B}(x) [c(t, t + d) + e^{-rd} N \max\{e^x, K\}] dx. \end{aligned} \quad (2.7)$$

Lastly, for $t + d > T$, the regulatory trigger would not occur since the time to maturity is less than the accumulated excursion time required for activating the Parisian trigger. In this case, the pricing formulation is much simplified since only the one-touch trigger remains. We may simply set $\tau_P = \infty$ in eq. (2.3) and perform the expectation calculation based on the knowledge of the joint density function of (τ_B, X_{τ_B}) .

There do not exist analytic formulas for the above conditional density functions of the first passage times. In this paper, we employ the extended Fortet method to derive the recursive schemes for calculating the numerical approximation of these conditional density functions (see Section 3). As part of the derivation procedure of the recursive Fortet schemes, it is necessary to derive an analytic formula of the distribution function of Y_t conditional on varying level of X_t that is defined as follows:

$$L(x, y, t; x_0, y_0, t_0)dx = Q(Y_t \leq y, X_t \in (x, x + dx) | X_{t_0} = x_0, Y_{t_0} = y_0).$$

Thanks to the joint Gaussian distribution of X_t and Y_t , it can be shown that

$$L(x, y, t; x_0, y_0, t_0) = \frac{e^{-\frac{[x-x_0 - (r-q-\frac{\sigma^2}{2})(t-t_0)]^2}{2\sigma^2(t-t_0)}}}{\sqrt{2\pi\sigma^2(t-t_0)}} N\left(\frac{y - \mu_Y}{\Sigma_Y}\right), \quad (2.8a)$$

where

$$\begin{aligned} \mu_Y &= y_0 e^{-\alpha(t-t_0)} + \mu [1 - e^{-\alpha(t-t_0)}] \\ &\quad + \frac{\rho\sigma\sigma_Y [1 - e^{-\alpha(t-t_0)}]}{\sigma^2(t-t_0)} \left[x - x_0 - \left(r - q - \frac{\sigma^2}{2} \right) (t - t_0) \right], \\ \Sigma_Y^2 &= \sigma^2(t-t_0) - \frac{\left\{ \frac{\rho\sigma\sigma_Y [1 - e^{-\alpha(t-t_0)}]}{\sigma^2(t-t_0)} \right\}^2}{\sigma^2(t-t_0)}. \end{aligned}$$

The derivation of eq. (2.8a) is presented in Appendix A. One can also obtain the analytic expression of an alternative distribution function defined as follows:

$$\begin{aligned} M(x, y, t; x_0, y_0, t_0)dx &= Q(Y_t > y, X_t \in (x, x + dx) | X_{t_0} = x_0, Y_{t_0} = y_0) \\ &= \frac{e^{-\frac{[x-x_0-(r-q-\frac{\sigma^2}{2})(t-t_0)]^2}{2\sigma^2(t-t_0)}}}{\sqrt{2\pi\sigma^2(t-t_0)}} \left[1 - N\left(\frac{y - \mu_Y}{\Sigma_Y}\right) \right]. \end{aligned} \quad (2.8b)$$

The earlier papers by Bernard *et al.* (2005, 2006, 2007) on pricing barrier options under stochastic interest rates employ the Fortet method to derive the density function of the first passage time to single barrier. Since we have the lower barrier B and upper barrier G in our CoCo pricing model, so it is necessary to derive two sets of equations for the individual first passage time to each of these two barriers. Let y_l and y_u be the respective lower barrier and upper barrier of the process Y_t , where $y_l < Y_{t_0} < y_u$; and let τ_{y_l} and τ_{y_u} denote the first passage time of Y_t hitting y_l and y_u , respectively. We define the following conditional joint density functions:

$$f_l(s, \xi) dsd\xi = Q(\tau_{y_l} \in (s, s + ds), \tau_{y_l} < \tau_{y_u}, X_{\tau_{y_l}} \in (\xi, \xi + d\xi) | X_{t_0} = x_0, Y_{t_0} = y_0);$$

$$f_u(s, \xi) dsd\xi = Q(\tau_{y_u} \in (s, s + ds), \tau_{y_u} < \tau_{y_l}, X_{\tau_{y_u}} \in (\xi, \xi + d\xi) | X_{t_0} = x_0, Y_{t_0} = y_0).$$

By the strong Markov property of the joint process of X_t and Y_t , and considering the disjoint events $\{\tau_{y_u} < \tau_{y_l}\}$ and $\{\tau_{y_l} < \tau_{y_u}\}$, we deduce the following pair of integral equations for the conditional first passage time density functions:

$$\begin{aligned} &L(x, y, t; x_0, y_0, t_0) \\ &= \int_{t_0}^t \int_{-\infty}^{\infty} f_l(s, \xi) L(x, y, t; \xi, y_l, s) d\xi ds + \int_{t_0}^t \int_{-\infty}^{\infty} f_u(s, \xi) L(x, y, t; \xi, y_u, s) d\xi ds; \end{aligned} \quad (2.9a)$$

and

$$\begin{aligned} &M(x, y, t; x_0, y_0, t_0) \\ &= \int_{t_0}^t \int_{-\infty}^{\infty} f_l(s, \xi) M(x, y, t; \xi, y_l, s) d\xi ds + \int_{t_0}^t \int_{-\infty}^{\infty} f_u(s, \xi) M(x, y, t; \xi, y_u, s) d\xi ds. \end{aligned} \quad (2.9b)$$

In the next section, we derive the recursive schemes for calculating the numerical approximation of the first passage time density functions by performing discretization of the above pair of integral equations. We then show how to use the approximate density function values to compute the fair value of the CoCo bond through numerical integration of the integral representation formula of the value functions shown in eqs. (2.6) and (2.7).

3 Construction of the numerical algorithm

The numerical algorithm for pricing the CoCo bond with both mechanical conversion (one-touch trigger) and regulatory conversion (Parisian trigger) involves the following sequential steps:

1. Determination of the numerical approximation of the density functions of the first passage times to the two barriers.
2. Calculation of the price functions $V_t(X_t, Y_t; T)$ and $V_t^G(X_t; T)$ where $t > T - d$. When the time to expiry is less than d , the Parisian trigger becomes immaterial.
3. Calculation of the price functions $V_t(X_t, Y_t; T)$ and $V_t^G(X_t; T)$ where $0 \leq t < T - d$, when both the mechanical conversion and regulatory conversion are under active mode.

As in most numerical algorithms, it is necessary to approximate the infinite domain of X_t by a finite computational domain $[x_{\min}, x_{\max}]$, where x_{\min} and x_{\max} are chosen such that the probability of X_t staying outside the finite domain $[x_{\min}, x_{\max}]$ is negligibly small. We discretize the computational domain of $[x_{\min}, x_{\max}] \times [0, T]$ by n_x spatial grids and n_T time steps, where $n_x \Delta x = x_{\max} - x_{\min}$ and $n_T \Delta t = T$. Here, Δx and ΔT are the spatial stepwidth and time step, respectively. Note that the grid points in the computational domain are characterized by (x_i, t_j) , where $x_i = x_{\min} + i \Delta x$, $i = 0, 1, \dots, n_x$ and $t_j = j \Delta t$, $j = 0, 1, \dots, n_T$.

First, we would like to derive the recursive schemes that compute the numerical approximation of the density/ distribution functions of the first passage times at varying values of the stock price. This is done by performing discretization of the pair of integral equations for the first passage time density functions. We define the following discrete grid values for the first passage time density/ distribution functions as numerical approximation of their continuous counterparts:

- (i) $q(m, i; y) \approx Q(\tau_y \in (t_m, t_m + \Delta t), X_{\tau_y} \in (x_i, x_i + \Delta x) | X_0, Y_0), \quad Y_0 > y;$
- (ii) $q^l(i; y) \approx Q(\tau_y > t_l, X_{t_l} \in (x_i, x_i + \Delta x) | X_0, Y_0), \quad t_l > 0;$
- (iii) $q_{y_1 < y_2}(m, i) \approx Q(\tau_{y_1} \in (t_m, t_m + \Delta t), \tau_{y_1} < \tau_{y_2}, X_{\tau_{y_1}} \in (x_i, x_i + \Delta x) | X_0, Y_0),$
 $Y_0 \in (y_1, y_2);$
- (iv) $q_{y_2 < y_1}(m, i) \approx Q(\tau_{y_2} \in (t_m, t_m + \Delta t), \tau_{y_2} < \tau_{y_1}, X_{\tau_{y_2}} \in (x_i, x_i + \Delta x) | X_0, Y_0),$
 $Y_0 \in (y_1, y_2);$
- (v) $q^d(i; y_1, y_2) \approx Q(\min(\tau_{y_1}, \tau_{y_2}) > d, X_d \in (x_i, x_i + \Delta x) | X_0, Y_0), \quad Y_0 \in (y_1, y_2).$

Here, we do not state explicitly the dependence of these numerical density and distribution functions on the initial values X_0 and Y_0 for notational simplicity.

Single-barrier discrete first passage time density and distribution

The first two discrete first passage time density and distribution function, $q(m, i; y)$ and $q^l(i; y)$, involve only single barrier y , so one may employ a similar approach of that of the recursive Fortet scheme proposed by Bernard *et al.* (2008). We take $t_0 = 0$ and assume $Y_0 > y$, and suppose Y_t falls below y at some later time $t_l > 0$, then Y_t must hit the barrier y at some time τ_y earlier than t_l . By the strong Markov property of the joint process X_t and

Y_t , and using the law of total probability, we obtain

$$\begin{aligned}
& L(x_i, y, t_l; x_0, y_0, t_0) dx \\
&= \int_0^{t_l} \int_{-\infty}^{\infty} Q(\tau_y = u, X_u = \xi | X_0 = x_0, Y_0 = y_0) \\
&\quad Q(Y_{t_l} < y, X_{t_l} \in (x_i, x_i + dx) | X_u = \xi, Y_u = y) d\xi du \\
&= \int_0^{t_l} \int_{-\infty}^{\infty} Q(\tau_y = u, X_u = \xi | X_0 = x_0, Y_0 = y_0) d\xi du [L(x_i, y, t_l; \xi, y, u) dx]; \tag{3.1}
\end{aligned}$$

By performing discretization of the above integral equation, we have

$$L(x_i, y, t_l; x_0, y_0, t_0) \approx \sum_{m=0}^l \sum_{k=0}^{n_x} q(m, k; y) L(x_i, y, t_l; x_k, y, t_m). \tag{3.2}$$

Recall that $L(x_i, y, t_l; x_0, y_0, t_0)$ admits closed form formula [see eq. (2.8a)]. It is desirable to derive a recursive scheme to determine $q(m, i; y)$ over successive time level m and at varying values of x_i .

At initiation, it is obvious that $q(0, i; y) = 0$ since $L(x_i, y, t_0; x_0, y_0, t_0) = 0$ for $y_0 \neq y$. Suppose we have obtained values for $q(m, i; y)$, $m = 0, 1, \dots, l$, we would like to derive a recursive scheme to determine $q(l+1, i; y)$ at varying values of i . By virtue of the discretized scheme in eq. (3.2), we deduce that

$$\begin{aligned}
L(x_i, y, t_{l+1}; x_0, y_0, t_0) &\approx \sum_{m=0}^l \sum_{k=0}^{n_x} q(m, k; y) L(x_i, y, t_{l+1}; x_k, y, t_m) \\
&\quad + \sum_{k=0}^{n_x} q(l+1, k; y) L(x_i, y, t_{l+1}; x_k, y, t_{l+1}).
\end{aligned}$$

Note that

$$L(x_i, y, t_{l+1}; x_k, y, t_{l+1}) = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{otherwise} \end{cases};$$

so we obtain the following recursive formula for $q(l+1, i; y)$:

$$q(l+1, i; y) = L(x_i, y, t_{l+1}; x_0, y_0, t_0) - \sum_{m=0}^l \sum_{k=0}^{n_x} q(m, k; y) L(x_i, y, t_{l+1}; x_k, y, t_m). \tag{3.3}$$

Next, we show how to determine $q^l(i; y)$ in terms of $q(m, i; y)$ and $M(x_i, y, t_l; x_0, y_0, t_0)$. Suppose $Y_0 > y$ while $Y_{t_l} > y$, then either Y_t hits the barrier y from above before time t_l or Y_t never hits the barrier between times 0 and t_l . Again, by the strong Markov property of the joint process of X_t and Y_t , conditioning on the first passage time τ_y , we obtain

$$\begin{aligned}
& M(x_i, y, t_l; x_0, y_0, t_0) dx \\
&= \int_0^{t_l} \int_{-\infty}^{\infty} Q(\tau_y = u, X_u = \xi | X_0 = x_0, Y_0 = y_0) \\
&\quad Q(Y_{t_l} > y, X_{t_l} \in (x_i, x_i + dx) | X_u = \xi, Y_u = y) d\xi du \\
&\quad + Q(\tau_y > t_l, X_{t_l} \in (x_i, x_i + dx) | X_0 = x_0, Y_0 = y_0). \tag{3.4}
\end{aligned}$$

Note that $\tau_y > t_l$ would imply $Y_{t_l} > y$ implicitly. By performing discretization of eq. (3.4), we obtain

$$q^l(i; y) = M(x_i, y, t_l; x_0, y_0, t_0) - \sum_{m=0}^l \sum_{k=0}^{n_x} q(m, k; y) M(x_i, y, t_l; x_k, y, t_m) \quad (3.5).$$

Two-barrier discrete first passage time density and distribution

To deal with the presence of two barriers in our CoCo pricing model, it is necessary to determine $q_{y_1 < y_2}(m, i)$, $q_{y_2 < y_1}(m, i)$ and $q^d(i; y_1, y_2)$. In a similar manner, we perform the discretization of eqs. (2.9a,b) as follows:

$$\begin{aligned} L(x_i, y_1, t_l; x_0, y_0, t_0) &= \sum_{m=0}^l \sum_{k=0}^{n_x} q_{y_1 < y_2}(m, k) L(x_i, y_1, t_l; x_k, y_1, t_m) \\ &\quad + \sum_{m=0}^l \sum_{k=0}^{n_x} q_{y_2 < y_1}(m, k) L(x_i, y_1, t_l; x_k, y_2, t_m); \\ M(x_i, y_2, t_l; x_0, y_0, t_0) &= \sum_{m=0}^l \sum_{k=0}^{n_x} q_{y_1 < y_2}(m, k) M(x_i, y_2, t_l; x_k, y_1, t_m) \\ &\quad + \sum_{m=0}^l \sum_{k=0}^{n_x} q_{y_2 < y_1}(m, k) M(x_i, y_2, t_l; x_k, y_2, t_m); \end{aligned}$$

where $y_0 \in (y_1, y_2)$.

Again, we would like to construct the recursive scheme that computes $q_{y_1 < y_2}(l, i)$ at successive time levels t_l , $l = 0, 1, \dots$ and varying values of x_i , $i = 0, 1, \dots, n_x$. At initiation, we deduce that

$$\begin{aligned} q_{y_1 < y_2}(0, i) &= L(x_i, y_1, t_0; x_0, y_0, t_0) = 0; \\ q_{y_2 < y_1}(0, i) &= M(x_i, y_2, t_0; x_0, y_0, t_0) = 0. \end{aligned}$$

Given the values of $q_{y_1 < y_2}(k, i)$ and $q_{y_2 < y_1}(k, i)$ for $k = 0, 1, \dots, m$, we would like to derive the recursive scheme that computes $q_{y_1 < y_2}(m+1, i)$ and $q_{y_2 < y_1}(m+1, i)$. By observing the following relations:

$$M(x_m, y_2, t_{l+1}; x_k, y_1, t_{l+1}) = 0$$

and

$$M(x_m, y_2, t_{l+1}; x_k, y_2, t_{l+1}) = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{otherwise} \end{cases};$$

and following a similar technique used in the single-barrier case, we obtain

$$\begin{aligned} q_{y_1 < y_2}(m+1, i) &= L(x_i, y_1, t_{m+1}; x_0, y_0, t_0) - \sum_{k=0}^m \sum_{j=0}^{n_x} q_{y_2 < y_1}(k, j) L(x_i, y_1, t_{m+1}; x_j, y_2, t_k) \\ &\quad - \sum_{k=0}^m \sum_{j=0}^{n_x} q_{y_1 < y_2}(k, j) L(x_i, y_1, t_{m+1}; x_j, y_1, t_k); \end{aligned} \quad (3.6a)$$

and

$$\begin{aligned}
q_{y_2 < y_1}(m+1, i) &= M(x_i, y_2, t_{m+1}; x_0, y_0, t_0) - \sum_{k=0}^m \sum_{j=0}^{n_x} q_{y_2 < y_1}(k, j) M(x_i, y_2, t_{m+1}; x_j, y_2, t_k) \\
&\quad - \sum_{k=0}^m \sum_{j=0}^{n_x} q_{y_1 < y_2}(k, j) M(x_i, y_2, t_{m+1}; x_j, y_1, t_k). \tag{3.6b}
\end{aligned}$$

Lastly, in order to derive a numerical approximation scheme for finding $q^d(i; y_1, y_2)$ where $Y_0 \in (y_1, y_2)$, we observe that the logarithm of the capital ratio Y_d at a later time d stays within (y_1, y_2) provided that (i) $Y_t \in (y_1, y_2)$ for $0 \leq t \leq d$, (ii) $\tau_{y_1} < \min\{\tau_{y_2}, d\}$ and $Y_d \in (y_1, y_2)$, (iii) $\tau_{y_2} < \min\{\tau_{y_1}, d\}$ and $Y_d \in (y_1, y_2)$. Again, by the strong Markov property of the joint process X_t and Y_t and using the law of total probability, we obtain

$$\begin{aligned}
&Q(Y_d \in (y_1, y_2), X_d \in (x_i, x_i + dx) | X_0, Y_0) \\
&= Q(\min\{\tau_{y_1}, \tau_{y_2}\} > d, X_d \in (x_i, x_i + dx) | X_0, Y_0) \\
&\quad + \int_0^d \int_{-\infty}^{\infty} Q(\tau_{y_1} = u, \tau_{y_1} < \tau_{y_2}, X_u = \xi | X_0, Y_0) \\
&\quad\quad Q(Y_d \in (y_1, y_2), X_d \in (x_i, x_i + dx) | X_u = \xi, Y_u = y_1) d\xi du \\
&\quad + \int_0^d \int_{-\infty}^{\infty} Q(\tau_{y_2} = u, \tau_{y_2} < \tau_{y_1}, X_u = \xi | X_0, Y_0) \\
&\quad\quad Q(Y_d \in (y_1, y_2), X_d \in (x_i, x_i + dx) | X_u = \xi, Y_u = y_2) d\xi du.
\end{aligned}$$

Let n_d be the number of time steps corresponding to the time period d , where $n_d \Delta t = d$. By discretization of the above equation, we obtain

$$\begin{aligned}
q^d(i; y_1, y_2) &= L(x_i, y_2, d; X_0, Y_0, t_0) - L(x_i, y_1, d; X_0, Y_0, t_0) \\
&\quad - \sum_{m=0}^{n_d} \sum_{j=0}^{n_x} q_{y_1 < y_2}(m, j) [L(x_i, y_2, d; x_j, y_1, t_m) - L(x_i, y_1, d; x_j, y_1, t_m)] \\
&\quad - \sum_{m=0}^{n_d} \sum_{j=0}^{n_x} q_{y_2 < y_1}(m, j) [L(x_i, y_2, d; x_j, y_2, t_m) - L(x_i, y_1, d; x_j, y_2, t_m)]. \tag{3.7}
\end{aligned}$$

Numerical approximation of the price functions

The price function $V_t(X_t, Y_t; T)$ can be represented in an integral form as shown in eq. (2.6), which involves the auxiliary value function $V_u^G(x; T)$, conditional joint density $f_G^-(u, x)$ and marginal density function $F_G^-(x)$. For $t + d < T$, the auxiliary value function $V_t^G(X_t; T)$ admits the integral representation [see eq. (2.7)] that involves the marginal density function $F_{G^+, B}(x)$ and conditional density functions $f_{G^+}(u, x)$ and $f_B(u, x)$.

Recall from eq. (2.4) that $V_t^G(X_t; T)$ and $V_t(X_t, Y_t; T)$ are related to each other by taking $Y_t \rightarrow \ln G - \varepsilon$, where $\varepsilon \rightarrow 0^+$, under the continuous time formulation. We assume that the regulatory authority would initiate monitoring of the issuing bank when the issuer's capital ratio falls slightly below G and the monitoring phase is terminated when the capital

ratio rises slightly above G . In our numerical algorithm, we choose a finite small positive constant ε such that the monitoring phase is initiated when Y_t moves down to $\ln G - \varepsilon$ and monitoring is terminated when Y_t rises above $\ln G + \varepsilon$. We are then interested to find numerical approximation to the price functions $V_t(X_t, \ln G + \varepsilon; T)$ and $V_t^G(X_t; T)$ in terms of the numerical approximation to the marginal density functions derived in the above. This is done by performing numerical integration of the integral formulas of $V_t(X_t, Y_t; T)$ and $V_t^G(X_t; T)$.

Since the Parisian trigger becomes ineffective when the time to maturity is less than d , or $t \geq T - d$, it is necessary to separate our numerical calculations in the two separate cases (i) $t \geq T - d$ and (ii) $0 \leq t < T - d$. In our subsequent exposition, we write $q(m, i; y)|_{X_0, Y_0}$, $q^l(i, y)|_{X_0, Y_0}$, etc. to specify the initial values for X_t and Y_t to be X_0 and Y_0 , respectively for better clarity.

(i) $t \geq T - d$

Suppose the discrete time level $t_m \in [T - d, T]$, we then have

$$\begin{aligned} V_{t_m}(x_i, \ln G + \varepsilon; T) &\approx \sum_{l=m}^{n_T} \sum_{j=0}^{n_x} q(l - m, j; \ln B)|_{x_i, \ln G + \varepsilon} [c(t_m, t_l) + e^{-r(t_l - t_m)} N \max\{e^{x_j}, K\}] \\ &\quad + \sum_{j=0}^{n_x} q^{n_T - m}(j; \ln B)|_{x_i, \ln G + \varepsilon} [c(t_m, T) + e^{-r(T - t_m)} F]; \end{aligned} \quad (3.8a)$$

and

$$\begin{aligned} V_{t_m}^G(x_i; T) &\approx \sum_{l=m}^{n_T} \sum_{j=0}^{n_x} q(l - m, j; \ln B)|_{x_i, \ln G - \varepsilon} [c(t_m, t_l) + e^{-r(t_l - t_m)} N \max\{e^{x_j}, K\}] \\ &\quad + \sum_{j=0}^{n_x} q^{n_T - m}(j; \ln B)|_{x_i, \ln G - \varepsilon} [c(t_m, T) + e^{-r(T - t_m)} F]. \end{aligned} \quad (3.8b)$$

(ii) $0 \leq t < T - d$

Suppose the time level $t_m \in [0, T - d]$, we have

$$\begin{aligned} &V_{t_m}(x_i, \ln G + \varepsilon; T) \\ &\approx \sum_{l=m}^{n_T} \sum_{j=0}^{n_x} q(l - m, j; \ln G - \varepsilon)|_{x_i, \ln G + \varepsilon} [c(t_m, t_l) + e^{-r(t_l - t_m)} V_{t_l}^G(x_j; T)] \\ &\quad + \sum_{j=0}^{n_x} q^{n_T - m}(j; \ln G - \varepsilon)|_{x_i, \ln G + \varepsilon} [c(t_m, T) + e^{-r(T - t_m)} F]; \end{aligned} \quad (3.9a)$$

and

$$\begin{aligned}
& V_{t_m}^G(x_i; T) \\
& \approx \sum_{l=m}^{m+n_d} \sum_{j=0}^{n_x} q_{\ln B < \ln G + \varepsilon}(l - m, j) |_{x_i, \ln G - \varepsilon} [c(t_m, t_l) + e^{-r(t_l - t_m)} N \max\{e^{x_j}, K\}] \\
& + \sum_{l=m}^{m+n_d} \sum_{j=0}^{n_x} q_{\ln G + \varepsilon < \ln B}(l - m, j) |_{x_i, \ln G - \varepsilon} [c(t_m, t_l) + e^{-r(t_l - t_m)} V_{t_l}(x_j, \ln G + \varepsilon; T)] \\
& + \sum_{j=0}^{n_x} q^d(j; \ln B, \ln G + \varepsilon) |_{x_i, \ln G - \varepsilon} [c(t_m, t_m + d) + e^{-rd} N \max\{e^{x_j}, K\}]. \quad (3.9b)
\end{aligned}$$

In particular, the numerical approximation to the price function at initiation $V_0(X_0, Y_0; T)$ is given by

$$\begin{aligned}
V_0(X_0, Y_0; T) & \approx \sum_{l=0}^{n_T} \sum_{j=0}^{n_x} q(l, j; \ln G - \varepsilon) |_{X_0, Y_0} [c(t_0, t_l) + e^{-rt_l} V_{t_l}^G(x_j; T)] \\
& + \sum_{j=0}^{n_x} q^{n_T}(j; \ln G - \varepsilon) |_{X_0, Y_0} [c(t_0, T) + e^{-r(T-t_0)} F]. \quad (3.10)
\end{aligned}$$

4 Numerical results

In this section, we present numerical experiments that examine accuracy of numerical results for the price function of the CoCo bond and analyze the order of convergence of the Fortet algorithm. We also compare the Fortet algorithm with the Monte Carlo simulation method with regard to accuracy and run time efficiency. Lastly, we examine pricing behavior of the CoCo bonds under various model parameter values.

4.1 Numerical accuracy, order of convergence and computational efficiency of the Fortet algorithm

First, we would like to examine the discretization errors arising from spatial and temporal discretization of the computational domain. The parameter values used in our numerical calculations were chosen to be: $r = 0.03$, $q = 0.025$, $\sigma = 0.25$, $\sigma_Y = 0.5$, $\rho = 0.3$, $\alpha = 1.0488$, $\mu = \ln 0.1$, $T = 2$, $F = 1000$, $c = 0.08$, $K = 2.5$, $d = \frac{15}{63}$, $N = 100$, $G = 0.075$, $B = 0.05$. In Table 1, we present the numerical values of the price function of the CoCo bond computed using varying number of grids n_x in the spatial x -domain and number of time steps n_T in the temporal domain. We observe that the change in numerical value of the price function is almost insignificant with respect to different choices of n_x . However, the choices of n_T have more significant impact on numerical accuracy of the price function since a fine resolution of the first passage time is required for good numerical accuracy. In terms of computational costs, the CPU time required increases roughly by a factor of 5 to 8 when n_x is doubled and a factor of 3 when n_T is doubled.

Number of time steps, n_T	Number of x -grids, n_x	Numerical value of price function	CPU time (in seconds)
252	50	1050.3103	20.511
	100	1050.0603	102.486
	200	1050.0550	638.760
504	50	1035.3322	64.916
	100	1035.1232	345.698
	200	1035.1224	2851.536

Table 1: Impact of number of x -grids n_x and time steps n_T in the Fortet algorithm on numerical accuracy of the price function and the CPU time. The CPU time required increases roughly by a factor of 5-8 when n_x is doubled and a factor of 3 when n_T is doubled.

Number of time steps, n_T	Numerical value of price function	Relative percentage error	Ratio of difference in values
252	1050.3103	3.3195%	
504	1035.3322	1.8461%	
1008	1025.9632	0.9245%	1.5987
2016	1020.8578	0.4222%	1.8351
4032	1018.2806	0.1687%	1.9810
8064	1017.0192	0.0446%	2.0431

Table 2: The difference between two consecutive iterations increases roughly by a factor of two when the number of time steps is halved, indicating first order temporal accuracy of the Fortet algorithm. The benchmark Monte Carlo simulation result using 10 million simulation paths is 1016.5654.

Next, we performed another set of numerical tests to examine the order of convergence of the Fortet algorithm with respect to the number of time steps n_T . The same set of parameter values in Table 1 are used to generate the numerical values of the price function in Table 2, except that n_x is fixed to be 50. The numerical value of the price function of 1016.5654 was obtained by performing 10 million simulation paths in Monte Carlo simulation (used as benchmark comparison for numerical accuracy). In Table 2, we observe that the ratio of the relative errors is very close to 2. This indicates that the extended Fortet algorithm achieves first order temporal accuracy.

4.2 Pricing behavior of the CoCo bonds

We would like to examine pricing behavior of the CoCo bonds under various model parameter values. We examine the dependence of the price function of the CoCo bond, $V_0(X_0, Y_0; T)$, on X_0 , Y_0 and T . In Figure 1, we show the plot of $V_0(X_0, Y_0; T)$ against X_0 . The parameter values used in the numerical calculations for the base case are: $r = 0.03$, $q = 0.025$, $\mu = \ln 0.1$, $\sigma = 0.25$, $\sigma_Y = 0.5$, $\rho = 0.3$, $\alpha = 1.0488$, $d = \frac{15}{63}$, $T = 2$, $F = 1000$, $c = 0.08$,

X_0	Y_0	d	Probability of Parisian trigger	Probability of mechanical trigger
2	$\ln 0.15$	$d = \frac{15}{63}$	0.0979	0.0567
2	$\ln 0.12$	$d = \frac{15}{63}$	0.1339	0.0762
2	$\ln 0.09$	$d = \frac{15}{63}$	0.2126	0.1183
1.5	$\ln 0.12$	$d = \frac{15}{63}$	0.1345	0.0765
2	$\ln 0.12$	$d = \frac{15}{63}$	0.1339	0.0762
2.5	$\ln 0.12$	$d = \frac{15}{63}$	0.1340	0.0762
2	$\ln 0.12$	$d = \frac{10}{63}$	0.2301	0.0427
2	$\ln 0.12$	$d = \frac{5}{63}$	0.3678	0.0058

Table 3: Probability of the Parisian trigger and mechanical trigger at varying level of X_0 , Y_0 and d .

$N = 100$, $K = 2.5$, $G = 0.075$, $B = 0.05$ and $Y_0 = \ln 0.12$. As expected, the price function increases monotonically with respect to X_0 . Since the CoCo bond has a relatively high bond component and low level of optionality in equity payoff, the impact of interest rate r on the price function is more significant compared to the stock price volatility σ and correlation coefficient ρ of the underlying joint process of X_t and Y_t . Since the trigger conditions for conversion are highly dependent on Y_t , the capital ratio volatility σ_Y has a stronger influence on the price function compared to the stock price volatility σ . Next, we show the plot of $V_0(X_0, Y_0; T)$ against Y_0 in Figure 2. The parameter values used in the numerical calculations for the base case are the same as those for Figure 1, except that X_0 is taken to be 2. The price function is monotonically increasing with respect to Y_0 , an expected result. It is seen that the mean reversion level μ has a stronger influence on the price function when compared with the mean reversion speed α . A lower mean reversion level μ and a high mean reversion speed α would decrease the value of the CoCo bond since the chance of equity conversion becomes higher.

The consecutive occupation time factor d is the length of the time period that the regulatory authority allows the issuing firm to stay in distressed state before activating the loss absorption mechanism. Since there has not yet a single case of regulatory trigger occurring in CoCo bonds, we cannot calibrate the parameter d based on actual case studies. However, it is instructive to analyze how d may impact the triggering mechanisms. We performed numerical experiments to examine the impact of X_0 , Y_0 and d on the respective probability of the Parisian trigger and mechanical trigger (see Table 3). The numerical results were obtained using Monte Carlo calculations with 16128 time steps and 1 million simulation paths. The parameter values used in the calculations for the base case are the same as those used for generating the plots in Figure 1. As revealed from the results in Table 3, the level of Y_0 and consecutive occupation time factor d have relatively strong influence on the probability of the Parisian trigger and mechanical trigger. When d is chosen to be relatively small, signifying stronger regulatory monitoring on financial state of the issuing bank, the probability of the Parisian trigger increases quite significantly while that of the mechanical trigger becomes smaller. As confirmed from the numerical results in Table 3, the stock price

level X_0 has almost insignificant impact on the probability of equity conversion. This is not surprising since equity conversion is expected to be mostly dependent on the capital ratio.

5 Conclusion

We propose a two-state pricing model of the CoCo bond using the joint process of the stock price and Tier 1 capital ratio. The accounting trigger of equity conversion into shares of the bond issuer is activated when the capital ratio hits a lower threshold value while the regulatory trigger is activated when the capital ratio stays under the nonviable state consecutively for a certain period of time. The dimensionality of the CoCo bond pricing model increases by one due to the inclusion of an extra Parisian state variable that models the regulatory trigger mechanism. In this paper, we construct an effective numerical algorithm that avoids adding an additional path dependent state variable for the Parisian feature of regulatory trigger. The design of the algorithm is based on the extended Fortet method that solves a pair of integral equations for the determination of the random first passage times of the capital ratio process to the upper Parisian threshold (warning threshold of nonviability) and the lower one-touch knock-out threshold. Dimension reduction in the pricing algorithm is achieved via the determination of the auxiliary CoCo bond function at the Tier 1 capital ratio that is slightly below the warning threshold of nonviability. As revealed from our numerical experiments, the extended Fortet method is seen to compete favorably well in terms of accuracy and run time efficiency when compared with the Monte Carlo simulation method. Pricing properties of the CoCo bond price function under various model parameter values are presented. The volatility parameters of the stock price process and capital ratio process have relatively small influence on the price function when compared with the level of interest rate, consistent with the dominant bond nature of the CoCo bond. The probability of the Parisian trigger and mechanical trigger are strongly dependent on the level of capital ratio and the length of the consecutive occupation time period staying within the nonviable state.

6 Reference

- Avdjiev, S., Kartasheva, A., Bogdanova, B., 2013. CoCos: a primer. *BIS Quarterly Review*, 43-56.
- Bernard, C., Le Courtois, O., Quittard-Pinon, F., 2005. Market value of life insurance contracts under stochastic interest rates and default risk. *Insurance: Mathematics and Economics*, **36**, 499-516.
- Bernard, C., Le Courtois, O., Quittard-Pinon, F., 2006. Development and pricing of a new participating contract. *North American Actuarial Journal*, **10(4)**, 179-195.
- Bernard, C., Le Courtois, O., Quittard-Pinon, F., 2008. Pricing derivatives with barriers in a stochastic interest rate environment. *Journal of Economic and Dynamic Control*, **32**, 2903-2938.

- Brigo, D., Garcia, J., Pede, N., 2015. CoCo bond pricing with credit and equity calibrated first-passage firm value models. *International Journal of Theoretical and Applied Finance*, **18(3)**, 1550015 (31 pages).
- Cheriditio, P., Xu, Z., 2015a. A reduced form CoCo model with deterministic conversion intensity. *Working paper of Princeton University*.
- Cheriditio, P., Xu, Z., 2015b. Pricing and hedging CoCos. *Working paper of Princeton University*.
- Chung, T.K., Kwok, Y.K., 2016. Enhanced equity-credit modeling for contingent convertibles. *Quantitative Finance*, **16(10)**, 1511-1527.
- Coculescu, D., Geman, H., Jeanblanc, M., 2008. Valuation of default-sensitive claims under imperfect information. *Finance and Stochastics* **12**, 195-218.
- Collin-Dufresne, P., Goldstein, R.S., 2001. Do credit spreads reflect stationary leverage ratios? *Journal of Finance*, **56 (5)**, 1929-1957.
- De Spiegeleer, J., Höcht, S., Marquet, I., Schoutens, W., 2017. CoCo bonds and implied CET1 volatility. *Quantitative Finance*, (online).
- De Spiegeleer, J., Schoutens, W., 2012. Pricing contingent convertibles: A derivatives approach. *Journal of Derivatives*, 27-36.
- Glasserman, P., Nouri, B., 2012. Contingent capital with a capital-ratio trigger. *Management Science*, **58(10)**, 1816-1833.
- Kwok, Y.K., 2011. Lattice tree methods for strongly path dependent options. *Encyclopedia of Quantitative Finance*, edited by Cont. R., John Wiley and Sons Ltd, Chichester, United Kingdom, 1022-1027.
- Longstaff, F., Schwartz, E., 1995. A simple approach to valuing risky fixed and floating rate debt. *Journal of Finance*, **50(3)**, 789-820.
- Makarov, R., Metzler, A., Ni, Z., 2015. Modeling default risk with occupation times. *Finance Research letters*, **13**, 54-65.
- Pelger, M., 2012. Contingent convertible bonds: pricing, dilution costs and efficient regulation. *Working paper of University of California at Berkeley*.
- Ritzema, B.P., 2015, Understanding additional Tier 1 CoCo bond prices using first-passage time models. *Working paper of Erasmus School of Economics*.
- Wilkens, S., Bethke, N., 2014. Contingent convertible (CoCo) bonds: A first empirical assessment of selected pricing models. *Financial Analysts Journal*, **70(2)**, 59-77.

Appendix A – Derivation of eq. (2.8a)

From the dynamic equation for X_t ,

$$dX_t = \left(r - q - \frac{\sigma^2}{2} \right) dt + \sigma \left(\sqrt{1 - \rho^2} dW_t^1 + \rho dW_t^2 \right),$$

we obtain

$$\begin{aligned} \mathbb{E}[X_t | X_{t_0} = x_0] &= x_0 + \left(r - q - \frac{\sigma^2}{2} \right) (t - t_0), \\ \text{var}(X_t | X_{t_0} = x_0) &= \sigma^2 (t - t_0). \end{aligned}$$

Similarly, the mean and variance of the following mean reversion process of Y_t , where

$$dY_t = \alpha(\mu - Y_t) dt + \sigma_Y dW_t^2,$$

are found to be

$$\begin{aligned} \mathbb{E}[Y_t | Y_{t_0} = y_0] &= y_0 e^{-\alpha(t-t_0)} + \mu [1 - e^{-\alpha(t-t_0)}], \\ \text{var}(Y_t | Y_{t_0} = y_0) &= \frac{\sigma_Y^2}{2\alpha} [1 - e^{-2\alpha(t-t_0)}]. \end{aligned}$$

As X_t and Y_t is a joint Gaussian process, by the Projection Theorem, the mean and variance of $Y_t | X_t$ are given by

$$\begin{aligned} \mu_{Y_t | X_t} &= \mathbb{E}[Y_t] + \frac{\text{cov}(X_t, Y_t)}{\text{var}(X_t)} (X_t - \mathbb{E}[X_t]), \\ \Sigma_{Y_t | X_t}^2 &= \text{var}(Y_t) - \frac{\text{cov}(X_t, Y_t)^2}{\text{var}(X_t)}. \end{aligned}$$

It is straightforward to obtain

$$\text{cov}(X_t, Y_t | X_{t_0} = x_0, Y_{t_0} = y_0) = \frac{\rho \sigma \sigma_Y}{\alpha} [1 - e^{-\alpha(t-t_0)}].$$

Finally, the distribution function of Y_t conditional on varying level of X_t is given by the product of the density function f_{X_t} and the conditional distribution function $F_{Y_t | X_t}$ as follows:

$$\begin{aligned} L(x, y, t; x_0, y_0, t_0) &= f_{X_t}(x; x_0) F_{Y_t | X_t}(y; y_0) \\ &= \frac{1}{\sqrt{2\pi \text{var}(X_t | X_{t_0})}} e^{-\frac{\{x - \mathbb{E}[X_t | X_{t_0} = x_0]\}^2}{2 \text{var}(X_t | X_{t_0})}} \int_{-\infty}^y \frac{1}{\sqrt{2\pi \Sigma_Y^2}} e^{-\frac{(\xi - \mu_Y)^2}{2 \Sigma_Y^2}} d\xi \\ &= \frac{e^{-\frac{[x - x_0 - (r - q - \frac{\sigma^2}{2})(t - t_0)]^2}{2\sigma^2(t - t_0)}}}{\sqrt{2\pi \sigma^2(t - t_0)}} N\left(\frac{y - \mu_Y}{\Sigma_Y}\right). \end{aligned}$$

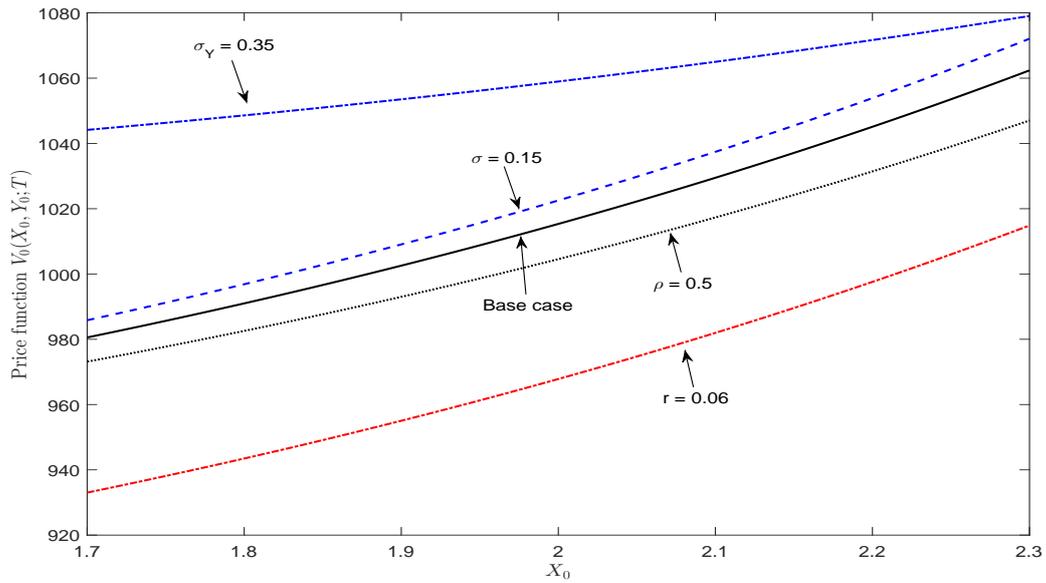


Figure 1: Plot of the price function $V_0(X_0, Y_0; T)$ against X_0 . The interest rate r and capital ratio volatility σ_Y are seen to have more significant impact on the price function when compared with the stock price volatility σ and correlation coefficient ρ of the joint process X_t and Y_t .

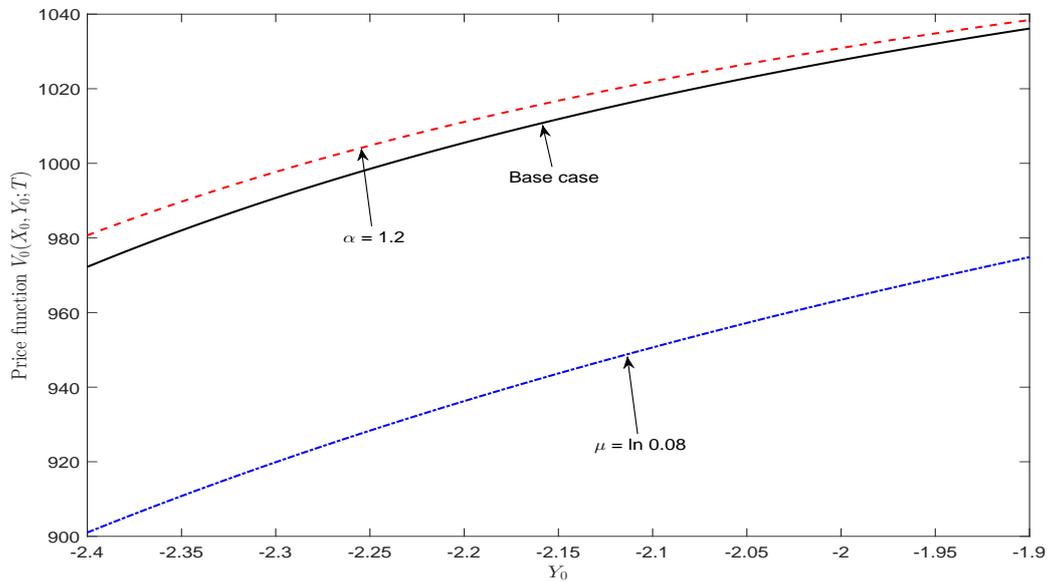


Figure 2: Plot of the price function $V_0(X_0, Y_0; T)$ against Y_0 . The mean reversion level of capital ratio μ shows stronger influence on the price function when compared with the mean reversion speed α .