Pricing Discrete Timer Options
under Stochastic Volatility Models

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Agenda

1. Product nature and uses of timer options
   - Barrier options in volatility space
   - Hedge downside risk with timing uncertainty / capture volatility premium

2. Analytic pricing of timer options under the stochastic volatility 3/2-model
   - Joint characteristic function of log-asset price and quadratic variation.

3. Numerical pricing of time options using Hilbert transform algorithm
   - The Fourier transform of a function multiplied by an indicator function (barrier feature) is related to the Hilbert transform of the Fourier transform
   - Evaluation via the Sinc expansion
Finite-maturity discrete timer options

• The price of a vanilla option is determined by the level of implied volatility quoted in the market. However, the level of implied volatility is often higher than the realized volatility, reflecting the risk premium due to the uncertainty of the future asset price movement.

• As revealed by empirical studies, 80% of the three-month calls that have matured in-the-money were overpriced.

• In a timer option, we fix volatility and allow maturity to float. This would resolve the volatility misspecification risk.
At the initiation of the timer option, the investor specifies an expected investment horizon $T_0$ and a target volatility $\sigma_0$ to define a variance budget

$$B = \sigma_0^2 T_0.$$  

Let $\tau_B$ be the first time in the tenor of monitoring dates at which the discrete realized variance exceeds the variance budget $B$, namely,

$$\tau_B = \min \left\{ j \mid \sum_{i=1}^{j} \left( \ln \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 \geq B \right\} \Delta.$$  

Here, $\Delta$ is the time interval between consecutive monitoring dates.

Termination date of a finite-maturity timer option $= \min(\tau_B, T)$, where $T$ is the preset mandated expiration date.
Uses of timer options

• Financial institutions can use the timer options to overcome the difficulty of pricing the call or put options whose implied volatility is difficult to quote.

• Portfolio managers can use timer put options on an index to limit their downside risk due to sudden market drops *(with uncertainty in timing)*.

• If one feels the implied volatility in the market is too high currently, then one can *capture the volatility risk* premium by longing a timer call and short a vanilla call. The volatility target is set below the current implied volatility and the volatility risk is captured by the difference in time value of the two options.
Partial differential equation for timer options

Suppose the stock price process $S_t$ and its instantaneous variance process $V_t$ satisfy

$$\frac{dS_t}{S_t} = r \, dt + \sqrt{V_t}(\sqrt{1 - \rho^2}dZ_t^1 + \rho \, dZ_t^2)$$
$$dV_t = \alpha(S_t, V_t) \, dt + \beta(S_t, V_t) \, dZ_t^2,$$

where $Z_t^1$ and $Z_t^2$ are uncorrelated standard Brownian motions under $Q$-measure.

Define the quadratic variation $I_t = \int_0^t V_s \, ds$. The price of a perpetual timer option $f(S_t, V_t, I_t)$ is governed by

$$V \frac{\partial f}{\partial I} + \alpha \frac{\partial f}{\partial V} + rS \frac{\partial f}{\partial S} + \frac{\beta^2}{2} \frac{\partial^2 f}{\partial V^2} + \frac{S^2V}{2} \frac{\partial^2 f}{\partial S^2} + \rho \beta S \sqrt{V} \frac{\partial^2 f}{\partial S \partial V} - rf = 0$$

with auxiliary condition: $f(S, V, B) = (S - K)^+$. 

Note that the variable $I_t$ plays the role of time.
Formal price formula of a timer call under the Heston model

In the Heston model, $\alpha(s) = k(\theta - s)$ and $\beta(s) = \gamma \sqrt{s}$. The joint distribution of $(\tau, V_\tau)$ is

$$(\tau, V_\tau) \overset{\text{Law}}{\sim} \left( \int_0^{B} \frac{1}{X_s} \, ds, X_B \right)$$

where $X_t$ is a Bessel process: $dX_t = \left( \frac{k\theta}{X_t} - k \right) \, dt + \gamma \, dW_t$, $X_0 = V_0$.

The price formula of a continuously monitored perpetual timer call option can be formally represented by

$$C_0 = S_0 E_Q[e^{\rho c(\tau, V_\tau)} - \frac{\rho^2}{2B} N(d_1(\tau, V_\tau))] - KE_Q[e^{-r\tau} N(d_2(\tau, V_\tau))]$$

where $\tau$ is the random termination date and

$$d_1(\tau, V_\tau) = \frac{\ln \frac{S_0}{K} + r\tau + \rho c(\tau, V_\tau) + \left( \frac{1}{2} - \rho^2 \right) B}{\sqrt{(1 - \rho^2)B}}$$

$$d_2(\tau, V_\tau) = d_1(\tau, V_\tau) - \sqrt{(1 - \rho^2)B}, \quad c(\tau, V_\tau) = \frac{V_\tau - V_0 - k\theta\tau + kB}{\gamma}.$$
Analytic pricing of discretely monitored finite-maturity timer options under the 3/2-model

We use the quadratic variation $I_t$ as a proxy of the discrete realized variance for the monitoring of the first hitting time. We define $\tau_B$ to be

$$\tau_B = \min \left\{ j \mid I_{t_j} \geq B \right\} \Delta.$$

This approximation does not introduce noticeable error for daily monitored timer options.

$$C_0(X_0, I_0, V_0) = \mathbb{E}_0[e^{-r(T \wedge \tau_B)} \max(S_{T \wedge \tau_B} - K, 0)]$$

$$= \mathbb{E}_0[e^{-rT} \max(S_T - K, 0)1_{\{\tau_B > T\}}] $$

$$+ e^{-r\tau_B} \max(S_{\tau_B} - K, 0)1_{\{\tau_B \leq T\}},$$

where $K$ is the strike price and $r$ is the constant interest rate.
Decomposition into a portfolio of timerlets

The event \( \{\tau_B > t\} \) is equivalent to \( \{I_t < B\} \). Note that \( \tau_B = t_{j+1} \) if and only if \( I_{t_j} < B \) and \( I_{t_{j+1}} \geq B \). Therefore, we have

\[
\{\tau_B \leq T\} = \bigcup_{j=0}^{N-1} \{I_{t_j} < B, I_{t_{j+1}} \geq B\}.
\]

The price of a finite-maturity discrete timer call option can be conveniently computed by decomposing it into a portfolio of timerlets as follows

\[
C_0 = \mathbb{E}_0\left[ e^{-rT} \max(S_T - K, 0) 1_{\{I_T < B\}} \right] \\
+ \mathbb{E}_0 \left[ \sum_{j=0}^{N-1} e^{-rt_{j+1}} \left( \max(S_{t_{j+1}} - K, 0) 1_{\{I_{t_j} < B\}} - \max(S_{t_{j+1}} - K, 0) 1_{\{I_{t_{j+1}} < B\}} \right) \right].
\]
The 3/2 stochastic volatility model

Consider the 3/2 stochastic volatility model specified as follows:

\[
\frac{dS_t}{S_t} = (r - q)dt + \sqrt{V_t} \left( \rho dW^1_t + \sqrt{1 - \rho^2} dW^2_t \right),
\]
\[
dV_t = V_t(\theta_t - \kappa V_t)dt + \varepsilon V_t^{3/2}dW^2_t,
\]

where \(W^1_t\) and \(W^2_t\) are two independent Brownian motions.

- The speed of mean reversion is now \(\kappa V_t\), which is linear in \(V_t\). The mean reversion is faster when the instantaneous variance is higher.
- \(\varepsilon\) cannot be interpreted as the same volatility of variance in the Heston model.
Partial Fourier transform of the triple joint density function under 3/2 model

Write \( X_t = \ln S_t \) and quadratic variation by \( I_t = \int_0^t V_s \, ds \), where \( I_t \) is used as a proxy for the discrete realized variance used in the barrier condition in a timer option.

Let \( G(t, x, y, v; t', x', y', v') \) be the joint transition density of the triple \((X, I, V)\) from state \((x, y, v)\) at time \( t \) to state \((x', y', v')\) at a later time \( t' \).

The joint transition density \( G \) satisfies the following three-dimensional Kolmogorov backward equation:

\[
- \frac{\partial G}{\partial t} = \left( r - q - \frac{v}{2} \right) \frac{\partial G}{\partial x} + \frac{v}{2} \frac{\partial^2 G}{\partial x^2} + v \frac{\partial G}{\partial y} + v(\theta_t - \kappa v) \frac{\partial G}{\partial v} + \varepsilon^2 v^3 \frac{\partial^2 G}{\partial v^2} + \rho \varepsilon v^2 \frac{\partial^2 G}{\partial x \partial v},
\]

with the terminal condition:

\[
G(t, x, y, v; t', x', y', v') = \delta(x - x')\delta(y - y')\delta(v - v'),
\]

where \( \delta(\cdot) \) is the Dirac delta function.
We define the generalized partial Fourier transform of $G$ by $\tilde{G}$ as follows:

$$
\tilde{G}(t, x, y, v; t', \omega, \eta, v') = \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{i\omega x' + i\eta y'} G(t, x, y, v; t', x', y', v') dy' dx',
$$

where the transform variables $\omega$ and $\eta$ can be complex variables. The partial transform $\tilde{G}$ also solves the above PDE with the terminal condition: $\tilde{G}(t', x, y, v; t', \omega, \eta, v') = e^{i\omega x + i\eta y} \delta(v - v').$

Note that $\tilde{G}$ admits the following solution form:

$$
\tilde{G}(t, x, y, v; t', \omega, \eta, v') = e^{i\omega x + i\eta y} g(t, v; t', \omega, \eta, v'),
$$

where $g$ satisfies the following PDE:

$$
-\frac{\partial g}{\partial t} = \left[ i\omega \left( r - q - \frac{v}{2} \right) - \omega^2 \frac{v}{2} + i\eta v \right] g + \left[ v(\theta_t - \kappa v) + i\omega \rho v^2 \right] \frac{\partial g}{\partial v} + \frac{\varepsilon^2 v^3}{2} \frac{\partial^2 g}{\partial v^2},
$$

with the terminal condition:

$$
g(t', v; t', \omega, \eta, v') = \delta(v - v').\]
The double generalized Fourier transform on the log-asset and integrated variance pair reduces the three-dimensional governing equation to a one-dimensional equation. We manage to obtain

\[ g(t, v; t', v') = e^{a(t'-t)} \frac{A_t}{C_t} \exp \left( - \frac{A_t v + v'}{C_t vv'} \right) (v') - 2 \left( \frac{A_t v}{v'} \right)^{\frac{1}{2} + \frac{\tilde{\kappa}}{\varepsilon^2}} I_{2c} \left( \frac{2}{C_t} \sqrt{\frac{A_t}{vv'}} \right), \]

where \( I_{2c} \) is the modified Bessel function of order \( 2c \),

\[ a = i\omega(r - q), \quad \tilde{\kappa} = \kappa - i\omega \rho \varepsilon, \quad A_t = e^{\int_t^{t'} \theta_s \, ds}, \]
\[ C_t = \frac{\varepsilon^2}{2} \int_t^{t'} e^{\int_t^s \theta_{s'} \, ds'} \, ds, \quad c = \sqrt{\left( \frac{1}{2} + \frac{\tilde{\kappa}}{\varepsilon^2} \right)^2 + \frac{i\omega + \omega^2 - 2\eta}{\varepsilon^2}}. \]

Note that \( c \) is in general complex and the numerical valuation of a modified Bessel function of complex order may pose some numerical challenge.
**Solution procedure for obtaining** \( g(t, v; t', v') \)

It is well known that the reciprocal of the 3/2 process is a CIR process. Indeed, if we define \( U_t = \frac{1}{V_t} \), then \( U_t \) is governed by

\[
dU_t = [(\kappa + \epsilon^2) - \theta_t U_t]dt - \epsilon \sqrt{U_t} dW_t.
\]

For any \( t' > t \), \( U_{t'} \) follows a non-central chi-square distribution conditional on \( U_t \). The corresponding (conditional) density function is given by

\[
p_U(U_{t'}|U_t) = \frac{A_t}{C_t} \exp \left( - \frac{A_t U_{t'} + U_t}{C_t} \right) \left( \frac{A_t U_{t'}}{U_t} \right)^{1+\frac{\kappa}{\epsilon^2}} I_{1+\frac{2\kappa}{\epsilon^2}} \left( \frac{2}{C_t} \sqrt{A_t U_{t'} U_t} \right).
\]
Evaluation of timelets

To evaluate the series of expectations in the timerlets, we derive the explicit representation for the characteristic functions of \((X_{t_j}, I_{t_j})\) and \((X_{t_j+1}, I_{t_j})\). Note that

\[
E_0[e^{i\omega X_{t_j} + i\eta I_{t_j}}] = e^{i\omega X_0 + i\eta I_0 h(t_0, V_0; t_j, \omega, \eta)},
\]

where

\[
h(t, v; t', \omega, \eta) = \int_0^{\infty} g(t, V_t; t', \omega, \eta, v') \, dv'
\]

\[
= e^{a(t'-t)} \frac{\Gamma(\bar{\beta} - \bar{\alpha})}{\Gamma(\bar{\beta})} \left( \frac{1}{C_{tv}} \right)^{\bar{\alpha}} M \left( \bar{\alpha}, \bar{\beta}, -\frac{1}{C_{tv}} \right),
\]

\[
\bar{\alpha} = -\frac{1}{2} - \frac{\bar{\kappa}}{\varepsilon^2} + c, \quad \bar{\beta} = 1 + 2c,
\]

\(\Gamma\) is the gamma function, \(M\) is the confluent hypergeometric function of the first kind.
The expectation calculation

$$\mathbb{E}_0[e^{-rt_{j+1}} \max (S_{t_{j+1}} - K, 0) \mathbf{1}_{\{I_{t_j} < B\}}]$$

requires the joint characteristic function at $t_j$ and $t_{j+1}$.

By virtue of a two-step expectation calculation, we obtain

$$\mathbb{E}_0[e^{i\omega X_{t_{j+1}} + i\eta I_{t_j}}] = e^{i\omega X_0 + i\eta I_0} \int_0^\infty g(t_0, V_0; t_j, \omega, \eta, v') h(t_j, v'; t_{j+1}, \omega, 0) \, dv'.$$

Here, $v'$ is the dummy variable for the variance $V_{t_j}$.

Working backward in time from $t_{j+1}$ to $t_j$, we compute $E_{t_j}[e^{i\omega X_{t_{j+1}}}]$ and from $t_j$ to $t_0$, we compute $E_0[e^{i\omega X_{t_j} + i\eta I_{t_j}}]$. This is done by setting $\eta = 0$ in $h(t_j, v'; t_{j+1}, \omega, 0)$ and integrating over $v'$ from 0 to $\infty$.

$$\mathbb{E}_0 \left[ e^{i\omega X_{t_j} + i\eta I_{t_j}} \right] \quad E_{t_j} \left[ e^{i\omega X_{t_{j+1}}} \right]$$
We transform the product of the terminal payoff and transition density function from the real domain to the Fourier domain via Parseval’s theorem.

**One-dimensional Parseval theorem**

\[
< f, g > = \int_{-\infty}^{\infty} f(x)g(x) \, dx = \frac{1}{2\pi} < \mathcal{F}f(u), \mathcal{F}g(u) >
\]

The Fourier transform of the terminal payoff \((S_{t_j+1} - K, 0)1_{\{t_j < B\}}\) and \((S_{t_j+1} - K, 0)1_{\{t_{j+1} < B\}}\) admit the same analytic representation

\[
\hat{F}(\omega, \eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega x - i\eta y} (e^x - K)^+1_{\{y < B\}} \, dx \, dy = \frac{K^{1-i\omega}e^{-i\eta B}}{(i\omega + \omega^2)i\eta},
\]

where \(x\) stands for \(\ln S_{t_j+1}\) and \(y\) stands for \(I_{t_j}\) or \(I_{t_{j+1}}\).

We take \(\omega = \omega_R + i\omega_I\) and \(\eta = \eta_R + i\eta_I\), where \(\omega_I < -1\) and \(\eta_I > 0\), to ensure the existence of the Fourier transform.
The finite-maturity discrete timer option price can be derived as follows

\[
C_0 = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-rT} \hat{F}(\omega, \eta) \mathbb{E}_0[e^{i\omega X_{t_N} + i\eta I_{t_N}}] \ d\omega_R d\eta_R \\
+ \sum_{j=0}^{N-1} \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-rt_{j+1}} \\
\left( \hat{F}(\omega, \eta) \mathbb{E}_0[e^{i\omega X_{t_{j+1}} + i\eta I_{t_{j+1}}}] - \hat{F}(\omega, \eta) \mathbb{E}_0[e^{i\omega X_{t_{j+1}} + i\eta I_{t_{j+1}}}] \right) \ d\omega_R d\eta_R \\
= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{F}(\omega, \eta) H(\omega, \eta) \ d\omega_R d\eta_R,
\]

where

\[
H(\omega, \eta) = e^{-rT} e^{i\omega X_0 + i\eta I_0} h(t_0, V_0; t_N, \omega, \eta) + e^{i\omega X_0 + i\eta I_0} \sum_{j=0}^{N-1} e^{-rt_{j+1}} \\
\left[ \int_{0}^{\infty} g(t_0, V_0; t_j, \omega, \eta, v') h(t_j, v'; t_{j+1}, \omega, 0) \ dv' - h(t_0, V_0; t_{j+1}, \omega, \eta) \right].
\]
Plot of finite-maturity discrete timer call option prices against variance budget $B$. The discrete timer call option price tends to that of the vanilla European call option (shown in the dashed line) when $B$ is sufficiently large.
Plot of finite-maturity discrete timer call option prices against number of monitoring instants $N$. The dashed line represents the finite-maturity timer call option price under continuous monitoring.
Plot of the finite-maturity discrete timer call option price versus maturity (mandated) under two different values of the variance budget.
Hilbert transform algorithm

The underlying asset price process $S_t$ and its instantaneous variance $v_t$ under a risk neutral measure $Q$ are assumed to follow the following general stochastic volatility model

$$\frac{dS_t}{S_t} = (r - q) \ dt + \sqrt{v_t} \ dW^S_t,$$
$$dv_t = \alpha(v_t) \ dt + \beta(v_t) \ dW^v_t,$$

where $E_t[dW^S_t dW^v_t] = \rho \ dt$. Here, $\rho$ is the correlation coefficient between the pair of Brownian motions, $q$ is the dividend yield, $\alpha(v_t)$ and $\beta(v_t)$ are measurable functions with respect to the natural filtration generated by the two correlated Brownian motions.

We define the continuous realized variance over $[0, t]$ by

$$I_t = \int_0^t v_s \ ds.$$
We define $\gamma_t = \ln V_t$, $x_t = \ln \frac{S_t}{K}$, where $K$ is the strike price, $\Delta$ be the uniform time interval between successive monitoring instants.

Let $V_{t_k}(x_{t_k}, \gamma_{t_k}, I_{t_k})$ be the option value of the finite-maturity discrete timer call option at monitoring time $t_k$, $k = 0, 1, \cdots, N$, where $x_{t_k}, \gamma_{t_k}$ and $I_{t_k}$ denote the time-$t_k$ normalized log-asset return, log-variance and realized variance, respectively.

Suppose we write the continuation value conditional on $\{I_k < B\}$ as

$$U_{t_k}(x_k, \gamma_k, I_k) = E_{t_k}[V_{t_{k+1}}(x_{k+1}, \gamma_{k+1}, I_{k+1})],$$

then we have

$$V_{t_k}(x_k, \gamma_k, I_k) = e^{-r\Delta}U_{t_k}(x_k, \gamma_k, I_k)1_{\{I_k < B\}} + K(e^{x_k} - 1)^+1_{\{I_k \geq B\}},$$

$k = 1, 2, \cdots, N - 1.$
We choose the log-variance instead of variance since the corresponding form of conditional density exhibits two advantages.

- The left tail of the conditional density of log-variance decays to zero more rapidly.

- The conditional densities of the log-variance processes for varying parameter values are more symmetric than those of the variance processes.
Time-stepping calculations between successive monitoring dates

By the tower property and conditional on the log-variance process $\gamma_{t_{k+1}}$ at time $t_{k+1}$, it follows that

$$U_{t_k}(x_k, \gamma_k, I_k) = E \left[ E \left[ V_{t_{k+1}}(x_{k+1}, \gamma_{k+1}, I_{k+1})|\mathcal{F}_{t_k}, \gamma_{k+1} \right] |\mathcal{F}_{t_k} \right].$$

The outer expectation integral involves integration over the density function $p_{\gamma}(\gamma_{t_{k+1}}|\gamma_{t_k})$, which has analytic closed form under the Heston and 3/2 stochastic volatility models.

- To evaluate the above three-dimensional expectation integral, we apply an interpolation based quadrature rule for the outer expectation integral and the Fourier transform method for the inner expectation integral.

- Across a monitoring date, as exemplified by the barrier condition, we can take advantage of the fast Hilbert transform method to deal with the barrier feature associated with the accumulated realized variance.
**FFT technique**

We adopt the numerical quadrature rule to calculate the outer expectation integral. By performing discretization along the dimension of log-variance $\gamma_{t_{k+1}}$ at the discrete nodes $\zeta_j, j = 1, 2, \ldots, J$, we obtain

$$U_{t_k}(x_k, \gamma_k, I_k) \approx \sum_{j=1}^{J} w_j p_{\gamma}(\zeta_j|\gamma_k) E \left[ V_{t_{k+1}}(x_{k+1}, \gamma_{k+1}, I_{k+1}) | F_{t_k}, \gamma_{k+1} = \zeta_j \right],$$

where $w_j$ is the weight at the quadrature node $\zeta_j, j = 1, 2, \ldots, J$.

We apply the Fourier transform method to perform the inner expectation calculations. To guarantee that the Fourier transforms are well defined, we need to introduce a proper exponential damping factor.

Let $w = \alpha_1 + i\beta_1$ and $u = \alpha_2 + i\beta_2$, where $\alpha_1$ and $\alpha_2$ are constants. At $\gamma_{t_{k+1}} = \zeta_j, x_{t_{k+1}} = x$ and $I_{t_{k+1}} = y$, we define

$$V_{t_{k+1}}^{\alpha_1, \alpha_2}(x, \zeta_j, y) = e^{\alpha_1 x + \alpha_2 y} V_{t_{k+1}}(x, \zeta_j, y).$$
The parameters $\alpha_1, \alpha_2$ are chosen to insure the existence of the generalized two-dimensional Fourier transform of $V_{t_{k+1}}(x, \zeta_j, y)$ as defined by

$$
\hat{V}_{t_{k+1}}^{\alpha_1, \alpha_2}(\zeta_j; \beta_1, \beta_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\beta_1 x + i\beta_2 y} V_{t_{k+1}}^{\alpha_1, \alpha_2}(x, \zeta_j, y) \, dx \, dy
$$

$$
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{w x + u y} V_{t_{k+1}}(x, \zeta_j, y) \, dx \, dy,
$$

where $w = \alpha_1 + i\beta_1$ and $u = \alpha_2 + i\beta_2$.

By the renowned Parseval’s theorem, we can represent the inner expectation as follows

$$
E[V_{t_{k+1}}(x_{k+1}, \gamma_{k+1}, I_{k+1})|\mathcal{F}_{t_k}, \gamma_{k+1} = \zeta_j] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V_{t_{k+1}}(x, \zeta_j, y) \, p(x, y|\mathcal{F}_{t_k}, \gamma_{k+1} = \zeta_j) \, dx \, dy
$$

$$
= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{V}_{t_{k+1}}^{\alpha_1, \alpha_2}(\zeta_j; \beta_1, \beta_2) \tilde{p}(w, u|\mathcal{F}_{t_k}, \gamma_{k+1} = \zeta_j) \, d\beta_1 \, d\beta_2.
$$
Here, $\tilde{p}(w, u|\mathcal{F}_{t_k}, \gamma_{k+1} = \zeta_j) = E\left[e^{-wx-uy} | \mathcal{F}_{t_k}, \gamma_{k+1} = \zeta_j \right]$ is visualized as the generalized inverse Fourier transform of the joint conditional density function $p(x, y|\mathcal{F}_{t_k}, \gamma_{k+1} = \zeta_j)$ of $x_{t_k+1}$ and $I_{t_k+1}$.

It is convenient to express $\tilde{p}(w, u|\mathcal{F}_{t_k}, \gamma_{k+1} = \zeta_j)$ in the following analytic representation

$$\tilde{p}(w, u|\mathcal{F}_{t_k}, \gamma_{k+1} = \zeta_j) = e^{-wx_k-uI_k} E\left[e^{-w(x_{k+1}-x_k)-u(I_{k+1}-I_k)} | \mathcal{F}_{t_k}, \gamma_{k+1} = \zeta_j \right].$$

Furthermore, we write $\Psi(w, u; \gamma_t, \gamma_s) = E\left[e^{w(x_t-x_s)+u(I_t-I_s)} | \mathcal{F}_s, \gamma_t \right]$. Here, we have suppress the dependency of $\Psi$ on $t - s$ for notational convenience.
By the tower property, for $s < t$, we have

$$
\Psi(w, u; \gamma_t, \gamma_s) = E \left[ E[e^{w(x_t-x_s)+u(I_t-I_s)}|F_s, \gamma_t, I_t - I_s]|F_s, \gamma_t \right] \\
= E \left[ E[e^{w(x_t-x_s)}|F_s, \gamma_t, I_t - I_s]e^{u(I_t-I_s)}|F_s, \gamma_t \right].
$$

Here, $\Psi(w, u; \gamma_t, \gamma_s)$ possesses closed form analytic representation under the Heston model and the 3/2 stochastic volatility model.

We may express the inner expectation integral at $\gamma_{t_k+1} = \zeta_j$ as shown in the following two-dimensional inverse Fourier transform representation

$$
E[V_{t_{k+1}}(x_{k+1}, \gamma_{k+1}, I_{k+1})|F_{t_k}, \gamma_{k+1} = \zeta_j] \\
= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-wx_k-uI_k} \hat{V}_{t_{k+1}}^{\alpha_1, \alpha_2}(\zeta_j; \beta_1, \beta_2) \Psi(-w, -u; \zeta_j, \gamma_k) \, d\beta_1 \, d\beta_2.
$$

Here, we have set $\gamma_{t_{k+1}} = \zeta_j$ in the conditional moment generating function $\Psi(w, u; \gamma_{k+1}, \gamma_k)$. 
Kernel functions for the Heston model and 3/2-model

The kernel function $\Psi(w,u; \gamma_t, \gamma_s)$ is the input that characterizes the specific stochastic volatility model. We derive the explicit representation of $\widetilde{\Psi}(w,u; \gamma_t, \gamma_s)$ for the Heston model and 3/2 stochastic volatility model.

Under the general stochastic volatility model, the asset price process admits the following representation

$$S_t = S_0 e^{(r-q)t + a_t + \sqrt{b_t} W},$$

where $a_t$ and $b_t$ are defined by

$$a_t = \rho [f(v_t) - f(v_0)] - \rho H_t - \frac{I_t}{2} \quad \text{and} \quad b_t = (1 - \rho^2) I_t,$$

with

$$H_t = \int_0^t h(v_s) \, ds \quad \text{and} \quad f(v_t) = \int \frac{\sqrt{v_t}}{\beta(v_t)} \, dv_t.$$

Here, $W$ is a standard normal random variable and $h$ is defined by

$$h(v_t) = \alpha(v_t) f'(v_t) + \frac{1}{2} \beta^2(v_t) f''(v_t).$$
**Heston model**

For the Heston stochastic volatility model, the dynamics for its variance is defined by

\[ d v_t = \lambda (\bar{v} - v_t) \, dt + \eta \sqrt{v_t} \, dW_t^v. \]

In the Heston model, \( \alpha(v_t) = \lambda (\bar{v} - v_t) \) and \( \beta(v_t) = \eta \sqrt{v_t} \). It follows that \( f(v_t) = \frac{v_t}{\eta} \) and \( h(v_t) = \frac{\lambda (\bar{v} - v_t)}{\eta} \).

We can rewrite the normalized log-asset return process as follows

\[ x_t = \ln \frac{S_0}{K} + (r - q) t + \frac{\rho}{\eta} (e^{\gamma t} - e^{\gamma 0} - \lambda \bar{v} t) + \left( \frac{\rho \lambda}{\eta} - \frac{1}{2} \right) I_t + \sqrt{1 - \rho^2} I_t W. \]
We obtain

\[ \Psi(w, u; \gamma_t, \gamma_s) = E\left[ E\left[ e^{w(x_t-x_s)}|F_s, \gamma_t, I_t - I_s \right] e^{u(I_t-I_s)}|F_s, \gamma_t \right] \]

\[ = e^w \left\{ (r-q)(t-s) + \frac{\rho}{\eta} e^{\gamma_t} - e^{\gamma_s} - \lambda \bar{v}(t-s) \right\} \]

\[ \Phi \left( -iw \left( \frac{\rho \lambda}{\eta} - \frac{1}{2} \right) - \frac{1}{2} iw^2 (1 - \rho^2) - iu; e^{\gamma_t}, e^{\gamma_s} \right), \]

where \( \Phi(\xi; \gamma_t, \gamma_s) = E[e^{i\xi \int_s^t v_u \, du} |\gamma_t, \gamma_s] \) is the conditional characteristic function of the time-integrated log-variance process \( \int_s^t v_u \, du \).
3/2 stochastic volatility model

The variance process evolves according to the following dynamics

\[ dv_t = \lambda v_t(\bar{v} - v_t) \, dt + \eta v_t^{3/2} \, dW_t. \]

The use of Itô’s formula gives the corresponding dynamics for \( \frac{1}{v_t} \)

\[ d \left( \frac{1}{v_t} \right) = \lambda \bar{v} \left( \frac{\lambda + \eta^2}{\lambda \bar{v}} - \frac{1}{v_t} \right) \, dt - \frac{\eta}{\sqrt{v_t}} \, dW_t. \]

The reciprocal of the variance process of the 3/2 model follows a mean-reverting square-root process with parameters \( (\lambda \bar{v}, \frac{\lambda + \eta^2}{\lambda \bar{v}}, -\eta) \).
In this case, \( \alpha(v_t) = \lambda v_t (\bar{v} - v_t) \) and \( \beta(v_t) = \eta v_t^{3/2} \). It follows that
\[
f(v_t) = \frac{\ln v_t}{\eta} \quad \text{and} \quad h(v_t) = \frac{\lambda}{\eta} \left[ \bar{v} - (1 + \frac{\eta^2}{2\lambda}) v_t \right].
\]

The normalized log-asset return process can be expressed as in the following form
\[
x_t = \ln \left( \frac{S_0}{K} \right) + (r-q) t + \frac{\rho}{\eta} [\gamma_t - \gamma_0 - \lambda \bar{v} t] + \left[ \frac{\rho \lambda}{\eta} \left( 1 + \frac{\eta^2}{2\lambda} \right) - \frac{1}{2} \right] I_t + \sqrt{(1 - \rho^2) I_t} W.
\]

Similarly, we have
\[
\Psi(w, u; \gamma_t, \gamma_s) = e^{w \left( (r-q)(t-s) + \frac{\rho}{\eta} [\gamma_t - \gamma_s - \lambda \bar{v}(t-s)] \right)}
\]
\[
\Phi \left( -iw \left[ \frac{\rho \lambda}{\eta} \left( 1 + \frac{\eta^2}{2\lambda} \right) - \frac{1}{2} \right] - \frac{1}{2} iw^2 (1 - \rho^2) - iu; e^{\gamma_t}, e^{\gamma_s} \right).
\]
Definition of Hilbert transform

For any $f \in L^p(\mathbb{R}), 1 \leq p < \infty$, we define the Fourier transform $\hat{f}$ by

$$\hat{f} = \mathcal{F}f = \int_{-\infty}^{\infty} e^{i\beta x} f(x) \, dx,$$

and $\hat{f} = \mathcal{F}f \in L^q(\mathbb{R})$ with $\frac{1}{p} + \frac{1}{q} = 1$.

For any $f \in L^p(\mathbb{R}), 1 \leq p < \infty$, its Hilbert transform is defined by the Cauchy principal value integral

$$\mathcal{H}f(x) = \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{f(y)}{x - y} \, dy,$$

and $\mathcal{H}f \in L^q(\mathbb{R})$ with $\frac{1}{p} + \frac{1}{q} = 1$. 
Two key properties of Hilbert transform

For any $b \in \mathbb{R}$, the Fourier transform of a function multiplied by an indicator function is related to the Hilbert transform of the Fourier transform function by

$$\mathcal{F}(1_{(-\infty,b)} \cdot f)(\beta) = \frac{1}{2}\hat{f}(\xi) - \frac{i}{2}e^{i\beta b}\mathcal{H}(e^{-i\beta b}\hat{f}(\eta))(\beta).$$

The Hilbert transform can be evaluated based on the Sinc expansion of an analytic function as follows

$$\mathcal{H}f(x) = \frac{1}{\pi}PV \int_{\mathbb{R}} \frac{f(y)}{x-y} \, dy = \sum_{l=-\infty}^{\infty} f(lh) \frac{1 - \cos \frac{\pi(x-lh)}{h}}{\pi(x-lh)\frac{h}{\pi}}, \quad h > 0,$$

where $h$ is the fixed discretization step.
Fast Hilbert transform algorithm

The backward induction procedure in the Fourier domain using the fast Hilbert transform algorithm for pricing finite-maturity discrete timer options can be reformulated as follows:

(i) We initiate our time stepping calculations at maturity $t_N$. The generalized Fourier transform of the terminal payoff admits the analytic formula

$$\hat{V}_{t_N}^{\alpha_1, \alpha_2}(\zeta_j; \beta_1, \beta_2) = -\frac{K}{(\alpha_1 + i\beta_1)(\alpha_1 + i\beta_1 + 1)(\alpha_2 + i\beta_2)},$$

for $j = 1, 2, \cdots, N$. Here, the constraints $\alpha_1 < -1$ and $\alpha_2 < 0$ should be observed in order to guarantee the existence of the above generalized Fourier transform.
(ii) For $k = N - 1, N - 2, \cdots, 1$, the numerical approximation of $\hat{V}_{t_k}^{\alpha_1, \alpha_2}(\zeta_p; \beta_1, \beta_2)$ is recursively calculated by computing a sequence of Hilbert transforms

$$
\hat{V}_{t_k}^{\alpha_1, \alpha_2}(\zeta_p; \beta_1, \beta_2) = e^{-r\Delta} \left[ \frac{1}{2} \hat{U}_{t_k}^{\alpha_1, \alpha_2}(\zeta_p; \beta_1, \beta_2) - \frac{i}{2} e^{i\beta_2 B} \mathcal{H} \left( e^{-i\beta_2' B} \hat{U}_{t_k}^{\alpha_1, \alpha_2}(\zeta_p; \beta_1, \beta_2') \right) (\beta_2) \right]
- Ke^{(\alpha_2+i\beta_2)B} \frac{1}{(\alpha_1+i\beta_1)(\alpha_1+i\beta_1+1)(\alpha_2+i\beta_2)}, \; p = 1, 2, \cdots, N.
$$
(iii) We approximate $\hat{U}_{t_k}^{\alpha_1,\alpha_2}(\zeta_p; \beta_1, \beta_2)$ using the quadrature rule

$$
\hat{U}_{t_k}^{\alpha_1,\alpha_2}(\zeta_p; \beta_1, \beta_2) \approx \sum_{j=1}^{J} w_j \hat{V}_{t_{k+1}}^{\alpha_1,\alpha_2}(\zeta_j; \beta_1, \beta_2) \tilde{\Psi}(-w, -u; \zeta_j, \zeta_p).
$$

(iv) For the last step where $k = 0$, the timer call option value is obtained by

$$
V_{t_0}(x_0, \zeta_p, I_0) 
\approx \sum_{j=1}^{J} \frac{e^{-r \Delta w_j}}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(\alpha_1+i\beta_1)x_0} e^{-(\alpha_2+i\beta_2)I_0} \hat{V}_{t_1}^{\alpha_1,\alpha_2}(\zeta_j; \beta_1, \beta_2) \tilde{\Psi}(-w, -u; \zeta_j, \zeta_p)
$$

for $p = 1, 2, \cdots, N$. 
**Sinc approximation**

Next, we provide the details on the evaluation of the Hilbert transform in the following form

\[ q(\beta) = \mathcal{H}\left(e^{-i\eta x} \hat{f}(\eta)\right)(\beta), \]

and the calculation of the two-dimensional inverse Fourier transform

\[
g(x_1, x_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\beta_1 x_1} e^{-i\beta_2 x_2} \hat{g}(\beta_1, \beta_2) \, d\beta_1 d\beta_2.
\]

The Hilbert transform can be evaluated by the truncated Sinc approximation

\[
q_{h,L}(\beta) = \sum_{l=-L}^{L} e^{-ilhx} \hat{f}(lh) \frac{1 - \cos \frac{\pi(\beta - lh)}{h}}{\pi(\beta - lh)}.
\]
The inverse Fourier transform can be evaluated numerically by the following discretized and truncated operator:

\[ g_{h_1,M,h_2,L}(x_1,x_2) = \frac{1}{4\pi^2} \sum_{m=-M}^{M} \sum_{l=-L}^{L} e^{-imh_1x_1}e^{-ilh_2x_2}\hat{g}(mh_1, lh_2)h_1h_2. \]

The trapezoidal sum approximation has been shown to be highly accurate, exhibiting exponentially decaying discretization errors.

- The overall computational complexity of the fast Hilbert transform algorithm for pricing finite-maturity discrete timer options is \( O(NM^2J^2L \log L) \), where \( N \) is the number of monitoring instants, \( M, J \) and \( L \) are the truncation level parameters in the log-asset dimension, log-variance dimension and realized variance dimension, respectively.
**Heston model**

<table>
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<tr>
<th>$S_0$</th>
<th>$T$</th>
<th>$r$</th>
<th>$q$</th>
<th>$B$</th>
<th>$N$</th>
<th>$\lambda$</th>
<th>$\eta$</th>
<th>$\bar{v}$</th>
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Parameter values in the Heston model and finite-maturity discrete timer options

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<th>$\rho$</th>
<th>Hilbert</th>
<th>MC</th>
<th>RE(%)</th>
</tr>
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Comparison of the numerical results for finite-maturity discrete timer call options for varying strike prices $K$ and correlation values $\rho$ obtained from the fast Hilbert transform algorithm with the benchmark results obtained using the Monte Carlo method (MC) under the Heston model.
Parameter values in the 3/2 model and finite-maturity discrete timer options

<table>
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<tr>
<th></th>
<th></th>
<th>Hilbert</th>
<th>MC</th>
<th>RE(%)</th>
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<td>-0.0774</td>
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Comparison of the numerical results for finite-maturity discrete timer call options for varying strike prices \( K \) and correlation values \( \rho \) obtained from the fast Hilbert transform algorithm with the benchmark results obtained using the Monte Carlo method (MC) under the 3/2 stochastic volatility model.
Sensitivity analysis on volatility of variance $\eta$ and correlation coefficient $\rho$ under the Heston model

- The price function may not be a monotonically increasing function of $\eta$.

- We observe that when $\rho = -0.5$, the discrete timer call option price firstly increases and then decreases with increasing value of $\eta$.

- On the other hand, when $\rho = 0.5$, the discrete timer call option price is a decreasing function of $\eta$. 
Comparison of the numerical values for finite-maturity discrete timer call option prices with varying values of strike prices, volatility of variance and correlation coefficient under the Heston model.

<table>
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<th>η</th>
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<th>K = 94</th>
<th>K = 98</th>
<th>K = 102</th>
<th>K = 106</th>
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</table>
Conclusion

• By decomposing a timer option into a portfolio of timerlets, we manage to price a finite-maturity timer option under the 3/2-model based on the explicit representation of the joint characteristic function of log asset price and its quadratic variation. The computational time for numerical evaluation of the joint characteristic function under the 3/2-model can be quite substantially lengthy.

• Our numerical tests on pricing finite-maturity discrete timer options under the Heston model and 3/2 model demonstrate high level of numerical accuracy and robustness of the fast Hilbert algorithm for pricing options with exotic barrier feature. The computational times required for pricing under different stochastic volatility models do not differ significantly.