Homework for Math 6050E: PDEs, Fall 2016

Tianling Jin

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Set 3

In this homework set, we always assume the coefficients of the various PDEs are smooth and satisfy the uniform ellipticity condition. Also, $\Omega \subset \mathbb{R}^n$ is always an open, bounded set with smooth boundary $\partial\Omega$.

Almost all the problems below are from Evans' book.

1. Consider the Laplacian equation with potential function $c(x)$:

$$-\Delta u + cu = 0, \tag{1}$$

and the equation in divergence form

$$-\operatorname{div}(a\nabla u) = 0, \tag{2}$$

where the function $a(x)$ is positive.
(a): Show that if u solves (1) and $w > 0$ also solves (1), then $v := u/w$ solves (2) for $a := w^2$.
(b): Conversely, show that if v solves (2), then $u := va^{1/2}$ solves (1) for some potential c.

2.A function $u \in H_0^2(\Omega)$ is a weak solution of this boundary value problem for the biharmonic equation

$$\begin{cases} \Delta^2 u & = f \quad \text{in } \Omega \\ u = \frac{\partial u}{\partial \nu} & = 0 \quad \text{on } \partial\Omega \end{cases} \tag{3}$$

provided

$$\int_\Omega \Delta u \Delta v dx = \int_\Omega f v dx \quad \text{for all } v \in H_0^2(\Omega).$$

Given $f \in L^2(\Omega)$, prove that there exists a unique weak solution of (3).

3. Assume Ω is connected. A function $u \in H^1(\Omega)$ is a weak solution of the Neumann's problem

$$\begin{cases} \Delta u & = f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} & = 0 \quad \text{on } \partial\Omega \end{cases} \tag{4}$$

if

$$\int_\Omega \nabla u \nabla v dx = \int_\Omega f v dx \quad \text{for all } v \in H^1(\Omega).$$

Suppose $f \in L^2(\Omega)$. Prove that (4) has a weak solution if and only if

$$\int_\Omega f dx = 0.$$

4. Let $u \in H^1(\mathbb{R}^n)$ have compact support and be a weak solution of the semilinear PDE

$$-\Delta u + c(u) = f \quad \text{in } \mathbb{R}^n,$$

where $f \in L^2(\mathbb{R}^n)$ and $c : \mathbb{R} \to \mathbb{R}$ is smooth with $c(0) = 0$ and $c' \geq 0$. Prove $u \in H^2(\mathbb{R}^n)$.

5. Let u be a smooth solution of $Lu := -\sum_{i,j=1}^n a^{ij} u_{ij} = 0$ in Ω. Assume all the coefficients a_{ij} are smooth and have bounded derivatives. Set $v := |\nabla u|^2 + \lambda u^2$. Show that $Lv \leq 0$ in Ω if λ is large enough. Then prove that

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C(\|\nabla u\|_{L^\infty(\partial\Omega)} + \|u\|_{L^\infty(\partial\Omega)}).$$

6. Assume Ω is connected. Use (a) energy methods and (b) the maximum principle to show that the only smooth solutions of the Neumann boundary value problem

$$\begin{cases} \Delta u & = 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} & = 0 \quad \text{on } \partial\Omega \end{cases}$$

are constant functions.

7. Assume $u \in H^1(\Omega)$ is a bounded weak solution of

$$-\sum_{i,j=1}^n (a^{ij} u_i)_j = 0 \quad \text{in } \Omega.$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex and smooth function. Set $w = \phi(u)$. Show that w is a weak subsolution, that is,

$$B[w, v] \leq 0 \quad \text{for all } v \in H^1(\Omega), v \geq 0.$$

8. We say that the uniformly elliptic operator

$$Lu := -\sum_{i,j=1}^n a^{ij} u_{ij} + b^i u_i + cu$$

satisfies the weak maximum principle if for all $u \in C^2(\Omega) \cap C(\overline{\Omega})$

$$\begin{cases} Lu & \leq 0 \quad \text{in } \Omega \\ u & \leq 0 \quad \text{on } \partial\Omega \end{cases}$$

implies that $u \le 0$ in Ω. Suppose that there exists a function $v \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $Lv \ge 0$ in Ω and $v > 0$ in Ω. Show that L satisfies the weak maximum principle. Note that we do NOT have sign assumption on c.

Hint: Find an elliptic operator M with no zeroth order term such that $w := u/v$ satisfies $Mw \le 0$ in the region $\{u > 0\}$. To do this, first compute $(v^2 w_i)_j$. See also the first problem here.

9. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy $\Delta u = 0$ in Ω. Assume that $u = \frac{\partial u}{\partial \nu} = 0$ on an open, smooth portion of $\partial\Omega$. Prove that u is identically zero.

Set 2

1. Let $0 < \alpha < 1$, $0 < \beta \le 1$ and $K > 0$ be constants. Let $u \in L^\infty([-1,1])$ satisfy $\|u\|_{L^\infty([-1,1])} \le K$. Define, for $h \in \mathbb{R}$ with $0 < |h| \le 1$,

$$v_{\beta,h}(x) = \frac{u(x+h) - u(x)}{|h|^\beta}, \quad x \in I_h,$$

where $I_h = [-1, 1-h]$ if $h > 0$ and $I_h = [-1-h, 1]$ if $h < 0$. Assume that $v_{\beta,h} \in C^\alpha(I_h)$ and $\|v_{\beta,h}\|_{C^\alpha(I_h)} \le K$ for every $0 < |h| \le 1$. Prove that

(i): If $\alpha + \beta < 1$ then $u \in C^{\alpha+\beta}([-1,1])$ and $\|u\|_{C^{\alpha+\beta}([-1,1])} \le CK$;

(ii): If $\alpha + \beta > 1$ then $u \in C^{0,1}([-1,1])$ and $\|u\|_{C^{0,1}([-1,1])} \le CK$,

where the constants C in (i) and (ii) depend only on $\alpha + \beta$.

2. Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose $u \in L^1_{loc}(\Omega)$ is a very weak solution of the Laplacian equation in the sense that

$$\int_\Omega u(x)\Delta\varphi(x)\mathrm{d}x = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

Prove that (up to redefinition on a set of measure zero) u is smooth in Ω and satisfies $\Delta u = 0$ pointwise in Ω.

Hint: use mollifiers to smooth u and pass to the limit.

3. Assume $0 < \beta < \gamma \le 1$ and Ω is an open set. Prove the interpolation inequality

$$\|u\|_{C^{0,\gamma}(\Omega)} \le \|u\|_{C^{0,\beta}(\Omega)}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C^{0,1}(\Omega)}^{\frac{\gamma-\beta}{1-\beta}}.$$

4. Prove directly that if $u \in W^{1,p}(0,1)$ for some $1 \le p < \infty$, where $(0,1)$ in the open interval on the real line, then u is equal a.e. to an absolutely contnuous function, and u' (which exists a.e.) belongs to $L^p(0,1)$.

5. Use integration by part to prove the interpolation inequality:

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \le C \left(\int_\Omega u^2 \mathrm{d}x \right)^{1/2} \left(\int_\Omega |\nabla^2 u|^2 \mathrm{d}x \right)^{1/2}$$

for all $u \in C_c^\infty(\Omega)$. Assume Ω is bounded and $\partial\Omega$ is smooth, prove this inequality for $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

6. Suppose Ω is open and connected, and $u \in W^{1,p}(\Omega)$ with $1 \le p \le \infty$ satisfies that

$$\nabla u = 0 \quad \text{in } \Omega.$$

Prove that u is constant in Ω.

7. (Chain rule) Assume $F : \mathbb{R} \to \mathbb{R}$ is C^1, with F' bounded. Suppose Ω is bounded and $u \in W^{1,p}(\Omega)$ for some $1 \le p \le \infty$. Show that

$$v := F(w) \in W^{1,p}(\Omega) \quad \text{and} \quad v_{x_i} = F'(u) u_{x_i},\ i = 1, \cdots, n.$$

8. Fix $\alpha > 0$ and let $\Omega = B(0,1)$ the unit ball centered at the origin. Show that there exists a constant C depending only on n, α such that

$$\int_\Omega u^2 \mathrm{d}x \le C \int_\Omega |\nabla u|^2 \mathrm{d}x$$

for all those $u \in H^1(\Omega)$ satisfying

$$|x \in \Omega : u(x) = 0| \ge \alpha.$$

9. Assume $1 \le p \le \infty$ and Ω is bounded and open. Let $u \in W^{1,p}(\Omega)$. Define $u^+ = \max(u, 0)$ and $u^- = -\min(u, 0)$. Prove that $u^+, u^-, |u| \in W^{1,p}(\Omega)$. Moreover,

$$\nabla u^+ = \begin{cases} \nabla u & \text{a.e. on } \{u > 0\} \\ 0 & \text{a.e. on } \{u \le 0\}, \end{cases}$$

$$\nabla u^- = \begin{cases} 0 & \text{a.e. on } \{u \ge 0\} \\ -\nabla u & \text{a.e. on } \{u < 0\}. \end{cases}$$

Also, prove that $\nabla u = 0$ a.e. on the set $\{u = 0\}$.

Note that the above problems 3-9 are from the main reference book: PDEs by L.C. Evans.

10. Let Ω be a non-empty open set in $\mathbb{R}^n$. Let $1 < p < \infty$. Prove that $W^{1,p}(\Omega)$ is reflexive.

11. (Hardy-Sobolev inequality) Let $n \ge 3$ and $s \in (0, 2]$. Prove that there exists a constant $C > 0$ depending only n and s such that

$$\left(\int_{\mathbb{R}^n} \frac{|u|^{\frac{2(n-s)}{n-2}}}{|x|^s} \right)^{\frac{n-2}{n-s}} \le C \int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x$$

for all $u \in C_c^\infty(\mathbb{R}^n)$.

Set 1

1. Write down an explicit formula for the solution of the following initial-value problem:

$$\begin{cases} u_t + b \cdot \nabla u + cu = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

2. Prove that Laplacian equation $\Delta u = 0$ is rotational invariant, that is, if O is an orthogonal $n \times n$ matrix and we define

$$v(x) = u(Ox) \quad x \in \mathbb{R}^n$$

then $\Delta v = 0$.

3. Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume that u is harmonic and $v := \phi(u)$. Prove that v is subharmonic. Also, prove that $w := |\nabla u|^2$ is subharmonic.

4. Let B be the unit ball centered at the origin in $\mathbb{R}^n$. Let u be a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } B \\ u = g & \text{on } \partial B. \end{cases}$$

Prove that there exists a positive constant C, which depends *only* on n, such that

$$\max_B |u| \leq C(\max_{\partial B} |g| + \max_B |f|).$$

5. Let B^+ denote the open half-ball $\{x \in \mathbb{R}^n : |x| < 1, x_n > 0\}$. Assume that $u \in C(\overline{B^+})$ is harmonic in B^+ and $u = 0$ on $\partial B^+ \cap \{x_n = 0\}$. For every $x \in B$, set

$$v(x) := \begin{cases} u(x) & \text{if } x_n > 0 \\ -u(x_1, \ldots, x_{n-1}, -x_n) & \text{if } x_n < 0. \end{cases}$$

Prove that v is harmonic in B.

Note that the above 5 problems are from the main reference book: PDEs by L.C. Evans.

6. Prove that every positive harmonic function in the whole space $\mathbb{R}^n$ has to be a constant function.

7. Let u be a harmonic function in an open set $\Omega \subset \mathbb{R}^n$ with $n \geq 3$. Let $\xi \in \mathbb{R}^n$ and $\lambda > 0$. Define

$$u_{\xi,\lambda}(x) := \left(\frac{\lambda}{|x - \xi|}\right)^{n-2} u\left(\xi + \frac{\lambda^2(x - \xi)}{|x - \xi|^2}\right).$$

This $u_{\xi,\lambda}$ is called the *Kelvin transform* of u. Prove that $u_{\xi,\lambda}$ is also harmonic in its domain.

8. Let B be the unit ball in $\mathbb{R}^n$ centered at the origin. Let u be a positive harmonic function in $B \setminus \{0\}$. Prove that there exist a harmonic function v in B and a constant $c \geq 0$ such that

$$u(x) = \begin{cases} c|x|^{2-n} + v(x), & \text{when } n \geq 3 \\ c|\log|x|| + v(x), & \text{when } n = 2 \end{cases} \qquad \text{for all } x \in B \setminus \{0\}.$$

This theorem can be stated as: Every positive harmonic function in the punctured ball with ***isolated singularity*** *has to be a fundamental solution plus a harmonic function in the whole ball.*