Homework for Math 6050F: PDEs, Spring 2017

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1. Consider the following Burgers' equation

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u(x, 0) = \sin(x) & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

(i). What is the first time T that the solution u becomes not C^1 in the x-variable?
(ii). What is the regularity of the function $u(x, T)$ (in the x-variable)?

The following problems are from Evans' book.

2. Compute explicitly the unique entropy solution of the Burgers' equation

$$\begin{cases} u_t + uu_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u(x, 0) = g(x) & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

where

$$g(x) = \begin{cases} 1, & x < -1 \\ 0, & -1 < x < 0 \\ 2, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

3. Find a solution of

$$-\Delta u + u^{\frac{n+2}{n-2}} = 0 \quad \text{in } B_1 := \{x : |x| < 1\}$$

having the form of $u = \alpha(1 - |x|^2)^{-\beta}$ for some positive constants α, β. This example shows that solutions of nonlinear PDE can be finite in the region, but approach to infinite on its boundary.

4. Show that there does not exists an analytic solution to the heat equation $u_t = u_{xx}$ in $\mathbb{R} \times \mathbb{R}$ with $u(x, 0) = \frac{1}{1+x^2}$.

5. Assume u is a viscosity solution of

$$u_t + H(Du, x) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

Show that $\hat{u} := -u$ is a viscosity solution of

$$\hat{u}_t + \hat{H}(D\hat{u}, x) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

for $\hat{H}(p,x) = -H(-p,x)$.

6. Let $\{u^k\}_{k=1}^{\infty}$ be viscosity solutions of the Hamilton-Jacobin equation

$$u_t^k + H(Du^k, x) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

Suppose that $u^k \to u$ uniformly, and H is continuous. Prove that u is also a viscosity solution of

$$u_t + H(Du, x) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

This says that the uniform limits of viscosity solutions are viscosity solutions.

7. Suppose for each $\varepsilon > 0$ that u^ε is a smooth solution of the viscous Hamilton-Jacobi equation

$$u_t^\varepsilon + H(Du^\varepsilon, x) - \varepsilon \sum_{i,j=1}^{n} a^{ij}(x,t) u^\varepsilon_{x_i x_j} = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty),$$

where a^{ij} are smooth and uniformly elliptic. Suppose that $u^\varepsilon \to u$ uniformly as $\varepsilon \to 0$ and H is continuous. Prove that u is a viscosity solution of

$$u_t + H(Du, x) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

This shows that the viscosity solutions do not depend on the precise smoothing structure.

8. Let $u^i (i = 1, 2)$ be bounded viscosity solutions of

$$\begin{cases} u_t^i + H(Du^i, x) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ u^i = g^i \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Assume that H satisfies

$$\begin{cases} |H(p,x) - H(q,x)| \le C|p-q| \\ |H(p,x) - H(p,y)| \le C|x-y|(1+|p|). \end{cases}$$

Prove the contraction property

$$\sup_{\mathbb{R}^n} |u^1(\cdot,t) - u^2(\cdot,t)| \le \sup_{\mathbb{R}^n} |g^1 - g^2| \quad (\forall\, t \ge 0).$$

9. (i) Show that $u(x) := 1 - |x|$ is a viscosity solution of

$$\begin{cases} |u'| = 1 \quad \text{in}(-1,1) \\ u(-1) = u(1) = 0. \end{cases} \tag{1}$$

This means that for each $v \in C^\infty(-1,1)$, if $u - v$ has a maximum (minimum) at a point $x_0 \in (-1,1)$, then $|v'(x_0)| \le 1 (\ge 1)$.

(ii) Prove that $\tilde{u}(x) = |x| - 1$ is not a viscosity solution of (1)

(iii) Show that $\tilde{u}$ is viscosity solution of

$$\begin{cases} -|u'| = -1 \quad \text{in } (-1,1) \\ u(-1) = u(1) = 0. \end{cases} \tag{2}$$

(What is the meaning of a viscosity solution of (2)?)

10. Let $\Omega \subset$ be open and bounded. For $x \in \Omega$, we define $u(x) := dist(x, \partial\Omega)$. Prove that u is Lipschitz continuous. Moreover, prove that u is a viscosity solution of the eikonal equation

$$|Du| = 1 \quad \text{in } \Omega.$$

This means that for each $v \in C^\infty(\Omega)$, if $u - v$ has a maximum (minimum) at a point $x_0 \in \Omega$, then $|Dv(x_0)| \leq 1 (\geq 1)$.

11. Suppose u is a smooth solution of

$$\begin{cases} u_t - \Delta u + c(x,t)u = 0 & \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(x,0) = g(x) & \text{on } \Omega \times \{t = 0\}. \end{cases} \tag{3}$$

Suppose the function g is bounded in Ω, and the function c satisfies $c \geq \gamma > 0$ in $\Omega \times (0,\infty)$. Prove

$$|u(x,t)| \leq Ce^{-\gamma t} \quad \text{in } \Omega \times (0,\infty)$$

12. Suppose u is a smooth solution of the PDE (3) from Problem 11. Suppose g is bounded and $g \geq 0$ in Ω, and c is bounded in $\Omega \times (0,\infty)$ (but not necessarily nonnegative). Prove that $u \geq 0$. (Hint: What PDE does $v := e^{\lambda t}u$ solve?)