Lecture notes on regularity of solutions of fully nonlinear elliptic equations

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1 Introduction

Let $0 < \lambda \leq \Lambda < +\infty$. For $M \in \mathcal{S} :=$ the set of $n \times n$ real symmetric matrices, we define

$$\mathcal{M}^-(M, \lambda, \Lambda) = \mathcal{M}^-(M) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i,$$
$$\mathcal{M}^+(M, \lambda, \Lambda) = \mathcal{M}^+(M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i,$$

where $e_i(M)$ are the eigenvalues of M. Note that suppose u is a (say, $C^{1,1}$) solution of

$$a_{ij}(x) u_{ij}(x) = c(x) \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is an open set, a_{ij} are merely measurable, bounded and satisfying the uniformly elliptic condition

$$\lambda I \leq a_{ij}(x) \leq \Lambda I \quad \text{in } \Omega,$$

and $c(x)$ satisfies $\|c\|_{L^\infty(\Omega)} \leq C_0$, then

$$\mathcal{M}^-(D^2 u) = \inf_{\lambda I \leq A \leq \Lambda I} tr(A D^2 u) \leq a_{ij}(x) u_{ij}(x) = c(x) \leq C_0 \quad \text{in } \Omega,$$

and

$$\mathcal{M}^+(D^2 u) = \sup_{\lambda I \leq A \leq \Lambda I} tr(A D^2 u) \geq a_{ij}(x) u_{ij}(x) = c(x) \geq -C_0 \quad \text{in } \Omega.$$

This note is devoted to regularity estimates for viscosity solutions of fully nonlinear elliptic equation

$$F(D^2 u, x) = 0 \quad \text{in } \Omega,$$

where F is uniformly elliptic, i.e., for every $M \in \mathcal{S}$ and every $x \in \Omega$

$$\lambda \|N\| \leq F(M+N,x) - F(M,x) \leq \Lambda \|N\| \quad \forall\, N \geq 0.$$

The regularity estimates we are going to show include: Harnack inequality, Hölder estimates, $C^{1,\alpha}$ estimates, the Evans-Krylov theorem, Schauder estimates and $W^{2,p}$ estimates.

- Harnack inequality and Hölder estimates as in the theory of Krylov and Safonov: their proofs will be presented by following those in [6] *without* using the ABP estimates, which is of more freedom, and can be adapted to certain *degenerate* elliptic equations [6]. Moreover, we will prove Hölder estimates *without* using the Harnack inequality. The method of proving the key growth lemma was used before in [2, 11].
- $C^{1,\alpha}$ estimates: their proof follows from the usual Jensen approximation and the difference quotient method as in [3], which is a quite standard technique as soon as we have the above Hölder estimate.
- the Evans-Krylov theorem: its proof will be presented as those in [4, 5], which can be adapted to nonlocal fully nonlinear equations in [5].
- Schauder estimates: their proof will be modified or re-organized compared to those in [3]. With extra work, it can be adapted to nonlocal fully nonlinear equation as in [7].

We say that a constant is universal if it depends only on the ellipticity constants λ, Λ and the dimension n. All the solutions in this note are understood in the viscosity sense.

2 Hölder regularity and Harnack inequality

Unlike the usual case, we will prove the following Hölder regularity without using the ABP estimates or the Harnack inequality.

Theorem 2.1. *Let u be a continuous function such that*

$$\begin{aligned} &\mathcal{M}^-(D^2u) \leq C_0 \quad \textit{in } B_1, \\ &\mathcal{M}^+(D^2u) \geq -C_0 \quad \textit{in } B_1, \\ &\|u\|_{L^\infty(B_1)} \leq C_0, \end{aligned}$$

then $u \in C^\alpha(B_{1/2})$, and there holds

$$\|u\|_{C^\alpha(B_{1/2})} \leq CC_0,$$

where $\alpha \in (0,1)$ and $C > 0$ are both universal constants.

Of course, we can also prove the Harnack inequality without using the ABP estimates.

Theorem 2.2. *Let u be a nonnegative continuous function such that*

$$\mathcal{M}^-(D^2u) \le C_0 \quad \textit{in } B_1,$$
$$\mathcal{M}^+(D^2u) \ge -C_0 \quad \textit{in } B_1$$

Then there holds

$$\sup_{B_{1/2}} u \le C(\inf_{B_{1/2}} u + C_0),$$

where $C > 0$ is a universal constant.

2.1 A growth lemma

Lemma 2.3. *There exists a small universal constant $\delta > 0$ such that for every lower semi-continuous function u satisfying*

$$u \ge 0 \quad \textit{in } B_1,$$
$$\mathcal{M}^-(D^2u) \le 1 \quad \textit{in } B_1,$$
$$|\{u > 2\} \cap B_1| > (1-\delta)|B_1|,$$

then $u > 1$ in $B_{1/4}$.

The proof of Lemma 2.3 is easier to understand when u is a smooth function. Thus, we will first describe the proof in this case, and then show how this proof works for lower semi-continuous viscosity super-solution in general.

Proposition 2.4. *Lemma 2.3 holds if we assume that $u \in C^2$.*

Proof. To prove Lemma 2.3, we only need to show that if

$$\begin{aligned} &u \ge 0 \quad \text{in } B_1,\\ &\mathcal{M}^-(D^2u) \le 1 \quad \text{in } B_1,\\ &\inf_{B_{1/4}} u \le 1, \end{aligned} \tag{2.1}$$

then

$$|\{u \le 2\} \cap B_1| \ge \delta|B_1|. \tag{2.2}$$

Let $x_0 \in B_{1/4}$ be such that $u(x_0) \le 1$. For every $x \in B_{1/4}$, let $y \in \overline{B}_1$ be a point where the minimum of $u(z) + 4|z-x|^2$, which is a function of z, is achieved. This is the same as that we slide the parabola $-4|z-x|^2$ from the below of u until they touch and y is a touch point. Note that

- When $z \in \partial B_1$, then $u(z) + 4|z-x|^2 \geq 0 + 4|1-1/4|^2 = 9/4$.
- $u(x_0) + 4|x_0 - x|^2 \leq 1 + 4|1/4+1/4|^2 = 2 < 9/4$.

Therefore, for every $x \in B_{1/4}$, such minimum point $y \in B_1$, and $u(y) + 4|y-x|^2 \leq u(x_0) + 4|x_0 - x|^2 \leq 2$. In particular, $u(y) \leq 2$. Note that for one value of x, there could be more than one point y where the minimum is achieved. However, the value of y uniquely determines x, since we must have

$$\nabla u(y) + 8(y-x) = 0, \quad D^2u(y) + 8I \geq 0. \tag{2.3}$$

Thus,

$$x = y + \frac{\nabla u(y)}{8}.$$

We define this as a map $x = m(y) = y + \frac{\nabla u(y)}{8}$, which is onto $B_{1/4}$. Consequently, we have that

$$\nabla m(y) = I + \frac{D^2u(y)}{8}$$

Since for each y we know $u(y) \leq 2$, the domain U of the map m satisfies that $U \subset \{y \in B_1 : u(y) \leq 2\}$. Thus, we have

$$|B_{1/4}| \leq \int_U |\det \nabla m(y)| \mathrm{d}y \leq \int_U |\det(I + \frac{D^2u(y)}{8})|$$

On the other hand, from the inequality $D^2u(y) + 8I \geq 0$ in (2.3) and the equation $\mathcal{M}^-(D^2u) \leq 1$ it follows that

$$|D^2u| \leq C$$

for some positive constant C depending only on λ, Λ, n. Thus, we have

$$|B_{1/4}| \leq C|U| \leq C|\{y \in B_1 : u(y) \leq 2\}|.$$

We can choose δ universally small such that (2.2) holds. □

Remark 2.5. *Here we slide a function, which is the parabola $-4|z-x|^2$ in uniformly elliptic case as in [11], from the below of u until they touch. The choice of this function is one freedom point of the proof, which may vary from cases to cases. In [2], the square of the distance function was used, while in [6], the cusp $-|z-x|^{1/2}$ was used.*

Proposition 2.6. *Lemma 2.3 holds if we assume that u is semi-concave.*

Proof. We say that u is semi-concave if $D^2u \leq A_0$ in the sense that $u(x) - A_0|x|^2/2$ is concave for some constant A_0. This means that for every point $x_0 \in B_1$ there exists a vector $p \in \mathbb{R}^n$ (a vector in the super-gradient set), which is $p = \nabla u(x_0)$ in case u is differentiable at x_0, so that

$$u(x) \leq u(x_0) + p \cdot (x - x_0) + \frac{A_0}{2}|x - x_0|^2. \tag{2.4}$$

for all $x \in B_1$. Basically, a function is semi-concave means that the function can be touched by a parabola from above.

We also recall that by Alexandrov theorem, the semi-concave function u is pointwise second differentiable almost everywhere. That means that there exists a set of measure zero $E \subset B_1$, so that at every point $x \in B_1 \setminus E$, the function u is differentiable and there exists a symmetric matrix, which is denoted as $D^2u(x)$, such that

$$u(y) = u(x) + (y - x) \cdot \nabla u(x) + \frac{1}{2}\langle D^2u(x)\,(y - x), (y - x)\rangle + o(|x - y|^2).$$

Moreover, we also have from [8, 10] that

$$\nabla u(y) = \nabla u(x) + D^2u(x)\,(y - x) + o(|x - y|),$$

where $\nabla u(y)$ is any vector in the super-gradient set of u at y.

We are going to show that if u is semi-concave and satisfies (2.1), then (2.2) holds. The proof will be similar to that of Proposition 2.4 with extra work dealing with semi-concavity instead of C^2.

Let $x_0 \in B_{1/4}$ be such that $u(x_0) \leq 1$. For every $x \in B_{1/4}$, let $y \in \overline{B}_1$ be a point where the minimum of $u(z) + 4|z - x|^2$, which is a function of z, is achieved. As before, for every $x \in B_{1/4}$, such minimum point $y \in B_1$, and $u(y) + 4|y - x|^2 \leq u(x_0) + 4|x_0 - x|^2 \leq 2$. In particular, $u(y) \leq 2$.

Since u is semi-concave, u can be touched by a parabola from above everywhere. At the point y, u is touched by the parabola $-4|z - x|^2 + \mathit{constant}$ at y from below. Therefore, u is differentiable at y. Consequently, we must have

$$\nabla u(y) + 8(y - x) = 0.$$

Thus,

$$x = y + \frac{\nabla u(y)}{8}.$$

We define this as a map $x = m(y) = y + \frac{\nabla u(y)}{8}$, which is onto $B_{1/4}$. Moreover, since for each y we know $u(y) \leq 2$, the domain U of the map m satisfies that $U \subset \{y \in B_1 : u(y) \leq 2\}$.

Again, note that at every point in U, the function u can be touched both from below and from above by two (uniform) parabola. From this it is elementary to check that ∇u is Lipschitz on U, and thus ∇u is differentiable a.e. in U.

Consequently, we have m is a Lipschitz map and it satisfies that

$$\nabla m(y) = I + \frac{D^2u(y)}{8} \quad \text{a.e. in } U.$$

Thus, we have

$$|B_{1/4}| \le \int_U |\det \nabla m(y)| \mathrm{d}y \le \int_U |\det(I + \frac{D^2u(y)}{8})|.$$

On the other hand, since y is a minimum point of $u(z) + 4|z-x|^2$ and ∇u is Lipschitz in U, we have

$$D^2u(y) + 8I \ge 0 \quad \text{a.e. in } U.$$

Together with the equation $\mathcal{M}^-(D^2u) \le 1$ it follows that

$$|D^2u| \le C \quad \text{a.e. in } U$$

for some positive constant C depending only on λ, Λ, n. Thus, we have

$$|B_{1/4}| \le C|U| \le C|\{y \in B_1 : u(y) \le 2\}|.$$

We can choose δ universally small such that (2.2) holds. □

Notice that in the above proof, we only use the property that u is semi-concave, and we didn't use the constant A_0 in (2.4). Now by Jensen approximation, we are able to prove Lemma 2.3.

Proof of Lemma 2.3. This time we only assume that u is a merely lower semi-continuous super-solution in B_1.

Let $v := \min(u, 4)$. Note that v is still a super-solution because it is the minimum of two super-solutions. We have $0 \le v \le 4$.

Consider the inf-convolution of v of parameter $\varepsilon > 0$:

$$v_\varepsilon(x) = \inf_{y \in B_1} (v(y) + (2\varepsilon)^{-1}|y-x|^2).$$

It is classical (here we may refer to [3]) to prove that v_ε is still a super-solution at $x \in B_{1-\eta}$ for $\eta > 0$ which will be determined as follows. Let $y_x \in \overline{B_1}$ be such that

$$v_\varepsilon(x) = v(y_x) + (2\varepsilon)^{-1}|y_x - x|^2 \le v(x).$$

Then

$$|y_x - x| \le 2\sqrt{\|v\|_\infty \varepsilon} = 4\sqrt{\varepsilon}. \tag{2.5}$$

Thus, for any $\eta > 0$, v_ε is a super-solution in $B_{1-\eta}$ provided that $4\sqrt{\varepsilon} < \eta$.

Note that

$$v(y_0) + (2\varepsilon)^{-1}|y_0 - x|^2 \le v_\varepsilon(x).$$

with equality holds for $x = 0$. Thus, v_ε is semi-concave.

Since v is lower semicontinuous, it is classical to show that v_ε Γ-converges to v, i.e.,

- for every sequence $x_k \to x$, $\liminf_{k\to\infty} v_k(x_k) \geq v(x)$;
- for every x there exists a sequence $x_k \to x$ such that $\limsup_{k\to\infty} v_k(x_k) = v(x)$;

We claim that with $\eta = 5\sqrt{\varepsilon}$,

$$\{x \in B_1 : v > 2\} = \bigcup_{\varepsilon>0} \{x \in B_{1-\delta} : v_\varepsilon > 2\}.$$

Indeed, on one hand, we know that $v \geq v_\varepsilon$, from which it follows that

$$\bigcup_{\varepsilon>0} \{x \in B_{1-\eta} : v_\varepsilon > 2\} \subset \{x \in B_1 : v > 2\}.$$

Now we are going to show that

$$\{x \in B_1 : v > 2\} \subset \bigcup_{\varepsilon>0} \{x \in B_{1-\eta} : v_\varepsilon > 2\}.$$

Let x be such that $v(x) > 2$. Then $v(x) > 2 + h$ for some $h > 0$. Also

$$v_\varepsilon(x) - v(x) \geq v(y_x) - v(x).$$

Since u is lower semi-continuous, v is lower semi-continuous as well. Then it follows from (2.5) that when ε is sufficiently small, we have

$$v(y_x) - v(x) > -h.$$

It follows that

$$v_\varepsilon(x) > v(x) - h > 2.$$

This finishes the claim.

Note that

$$\{x \in B_1 : u > 2\} = \{x \in B_1 : v > 2\},$$

and as $\varepsilon \to 0$, the sets $\{x \in B_{1-\eta} : v_\varepsilon > 2\}$ is an increasing nested collection. Therefore

$$|\{x \in B_1 : u > 2\}| = |\{x \in B_1 : v > 2\}| = \lim_{\varepsilon\to 0} |\{x \in B_{1-\eta} : v_\varepsilon > 2\}|.$$

For ε sufficiently small, we can apply Proposition 2.6 (appropriately scaled to the ball $B_{1-\eta}$ instead of B_1) and obtain that $v_\varepsilon \geq 1$ in $B_{(1-\eta)/4}$. Since $u \geq v_\varepsilon$ and η is arbitrarily small, the proof is finished.

□

2.2 A doubling property

Consider the barrier function $b(x) = |x|^{-p}$. We compute, for $x \in B_2 \setminus \{0\}$,

$$\begin{aligned}\mathcal{M}^-(D^2 b) &= \lambda p(p+1)|x|^{-p-2} - \Lambda(d-1)p|x|^{-p-2} \\ &= p|x|^{-p-2}\left(\lambda(p+1) - \Lambda(d-1)\right) \\ &\geq p|x|^{-p-2} \quad \text{if } p \text{ is large enough.}\end{aligned}$$

Thus, the function $b(x) = |x|^{-p}$ is a sub-solution of the Pucci equation $\mathcal{M}^-(D^2 b) \geq p2^{-p-2}$ in $B_2 \setminus \{0\}$.

Using this barrier function, we prove the following doubling property for lower bounds of super-solutions.

Lemma 2.7 (Doubling property for super-solutions)**.** *There exists a large universal constant $M_1 > 1$ such that if u is a nonnegative lower semi-continuous function satisfying $\mathcal{M}^-(D^2 u) \leq 1$ in B_2 and $u > M_1$ in $B_{1/4}$, then $u > 1$ in B_1.*

Proof. We compare the function u with

$$\varphi(x) := M_1 \frac{(|x|^{-p} - 2^{-p})}{2 \cdot 4^p}.$$

We choose $M_1 \geq 1$ sufficiently large so that $\varphi > 1$ in $B_{3/2}$. In $B_2 \setminus \{0\}$, we have

$$\mathcal{M}^-(D^2\varphi) = \frac{M_1}{2 \cdot 4^p}\mathcal{M}^-(D^2 b) \geq \frac{M_1}{2 \cdot 4^p} p2^{-p-2} \geq 2 \quad \text{for } M \text{ large enough.}$$

Moreover, $\varphi = 0$ on ∂B_2 and $\varphi < M_1$ in $\partial B_{1/4}$. Therefore, $\varphi \leq u$ in $B_2 \setminus B_{1/4}$ (this is the comparison principle between the viscosity super-solution u and the classical sub-solution B, which follows directly from the definition of viscosity solution).

Therefore, we have $u > 1$ in $B_{3/2}$. □

Combining Lemma 2.3 and Lemma 2.7, we obtain the following corollary

Corollary 2.8. *There exists a small universal constant $\delta > 0$ and a large universal constant $M > 0$ so that for every lower semi-continuous function $u : B_2 \to \mathbb{R}$ satisfying*

$$\begin{aligned}&u \geq 0 \textit{ in } B_2, \\ &\mathcal{M}^-(D^2 u) \leq 1 \textit{ in } B_2, \\ &|\{u > M\} \cap B_1| > (1-\delta)|B_1|,\end{aligned}$$

we have $u > 1$ in $B_{3/2}$.

Proof. Let $M_1 \geq 1$ be the one in Lemma 2.7 and let $M = 2M_1$. Then the function $v = u/M_1$ satisfies the assumption of Lemma 2.3. We conclude that $v > 1$ in $B_{1/4}$, *i.e.* $u > M_1$ in $B_{1/4}$. We then can apply Lemma 2.7 to obtain $u > 1$ in $B_{3/2}$. □

The following corollary is just a scaled version of the above result.

Corollary 2.9. *There exists a small universal constant $\delta > 0$ and a large universal constant $M > 0$ so that for every $r \leq 1$, every $\kappa \geq 1$ and every lower semi-continuous function $u : \overline{B_r} \to \mathbb{R}$ such that*

$$u \geq 0 \text{ in } B_{2r},$$
$$\mathcal{M}^-(D^2 u) \leq \kappa \text{ in } B_{2r},$$
$$|\{u > \kappa M\} \cap B_r| > (1-\delta)|B_r|,$$

then $u > \kappa$ in $B_{3r/2}$.

Proof. The scaled function $u_r(x) = u(rx)/\kappa$ satisfies the scaled equation

$$\mathcal{M}^-(D^2 u_r) \leq r^2 \leq 1 \text{ in } B_2.$$

Moreover, $u_r \geq 0$ in B_2 and $|\{u_r > M\} \cap B_1| > (1-\delta)|B_1|$. So we can apply Corollary 2.8 to u_r and have $u_r > 1$ in $B_{3/2}$. Hence, $u > \kappa$ in $B_{3r/2}$. □

2.3 The L^ε estimate

To obtain the so-called L^ε estimate, we first need the following growing ink-spots which was first introduced by E.M. Landis. We will prove it by using Vitali's covering lemma instead of the usual Caldéron-Zygmund decomposition.

Lemma 2.10 (Growing ink-spots lemma)**.** *Let $E \subset F \subset B_1$ be two open sets. Suppose the following two assumptions hold for some constant $\delta \in (0,1)$:*

- *$|E| \leq (1-\delta)|B_1|$. (This means that there is room for E to grow.)*
- *If any ball $B \subset B_1$ satisfies $|B \cap E| > (1-\delta)|B|$, then $B \subset F$. (This is a way that how E grows to F.)*

Then $|E| \leq (1-c\delta)|F|$ for some constant c depending on n only. Indeed, $c = 5^{-n}$ will do.

Proof. For every $x \in F$, since F is open, we define

$$r^x = \sup\{r > 0 : B_r(x) \subset F.\}$$

There exists some maximal ball, called B^x, which is contained in F and contains $B_{r^x}(x)$. This means that for any ball B such that $B^x \subset B \subset F$ then there holds $B^x = B$. One way to choose such a maximal ball is that if we let

$$R = \sup\{r : \text{there is a ball } D_r \text{ of radius } r \text{ such that } B_{r^x}(x) \subset D_r \subset F\},$$

then that ball of radius R is one maximal ball. We choose one of those maximal balls for each $x \in F$.

If $B^x = B_1$ for some $x \in F$, then the result of the theorem follows immediately since $|E| \leq (1-\delta)|B_1|$, so let us assume that it is not the case.

We claim that $|B^x \cap E| \leq (1-\delta)|B^x|$. Otherwise, we could find a slightly larger ball $\tilde{B}$ containing B^x such that $|\tilde{B} \cap E| > (1-\delta)|\tilde{B}|$ and $\tilde{B} \not\subset F$, contradicting the first hypothesis.

The family of balls B^x covers the set F. By the *Vitali covering lemma*, we can select a subcollection of at most countable disjoint balls $B_j := B^{x_j}$ such that $F \subset \bigcup_{j=1}^{K} 5B_j$, where $K \in \mathbb{N} \cup \{\infty\}$.

By construction, $B_j \subset F$ and $|B_j \cap E| \leq (1-\delta)|B_j|$. Thus, we have that $|B_j \cap F \setminus E| \geq \delta|B|$. Therefore

$$|F \setminus E| \geq \sum_{j=1}^{K} |B_j \cap F \setminus E| \geq \sum_{j=1}^{K} \delta|B_j| = \frac{\delta}{5^n} \sum_{j=1}^{K} |5B_j| \geq \frac{\delta}{5^n}|F|.$$

The proof is finished with $c = 1/5^n$. □

Combining Corollary 2.8 with Lemma 2.10, we obtain the L^ε estimate.

Theorem 2.11 (L^ε estimate)**.** *There exists a small universal constant $\varepsilon > 0$ and a large universal constant $C > 0$ so that for every lower semi-continuous function $u : B_2 \to \mathbb{R}$ such that*

$$\begin{aligned} &u \geq 0 \textit{ in } B_2, \\ &\mathcal{M}^-(D^2u) \leq 1 \textit{ in } B_2, \\ &\inf_{B_{3/2}} u \leq 1, \end{aligned}$$

then

$$|\{u \geq t\} \cap B_1| \leq Ct^{-\varepsilon}$$

for all $t > 0$.

Proof. In order to prove the result, we will prove the equivalent expression

$$|\{u > M^k\} \cap B_1| \leq \tilde{C}M^{-\varepsilon k}, \tag{2.6}$$

where M is the constant from Corollary 2.9 and $\varepsilon > 0$ will be properly chosen.

Let $A_k := \{u > M^k\} \cap B_1$, which are open sets. Since $\inf_{B_{3/2}} u \le 1$, from Corollary 2.8, $|A_1| \le (1-\delta)|B_1|$. Since $A_k \subset A_1$ for all $k > 1$, then we also have $|A_k| \le (1-\delta)|B_1|$ for all k.

We note that Corollary 2.9, with $\kappa = M^k$, says that every time a ball $B \subset B_1$ satisfies $|B \cap A_{k+1}| > (1-\delta)|B|$, then $B \subset A_k$. Using Lemma2.10 with $c = 5^{-n}$, we obtain

$$|A_{k+1}| \le (1 - c\delta)|A_k|,$$

and therefore, by induction, $|A_k| \le (1-c\delta)^{k-1}(1-\delta)|B_1| = \tilde{C}M^{-\varepsilon k}$, where $-\varepsilon = \log(1 - c\delta)/\log M$ and $\tilde{C} = (1-c\delta)^{-1}(1-\delta)|B_1|$.

For all $t > 1$, there exists k such that $M^k < t \le M^{k+1}$. Then from (2.6) we have

$$|\{u \ge t\} \cap B_1| \le |\{u > M^k\} \cap B_1| \le \tilde{C}M^{-\varepsilon k} \le \tilde{C}M^{\varepsilon}M^{-\varepsilon(k+1)} \le \tilde{C}Mt^{-\varepsilon}.$$

On the other hand, the conclusion for $t \le 1$ is trivial. This finishes the proof. □

The following is the rescaled L^ε estimate.

Lemma 2.12. *There exists a small universal constant $\varepsilon_1 > 0$ so that for every $r \le 1$, $\theta \ge r^2$, and every lower semi-continuous function $u : \overline{B_{2r}} \to \mathbb{R}$ such that*

$$\begin{aligned}
&u \ge 0 \text{ in } B_{2r},\\
&\mathcal{M}^-(D^2 u) \le \varepsilon_1 \text{ in } B_{2r},\\
&|\{u \ge \theta\} \cap B_r| \ge \frac{1}{2}|B_r|,
\end{aligned}$$

then $u > \varepsilon_1 \theta$ in $B_{3r/2}$.

Proof. Let τ be a universal large constant such that $C\tau^{-\varepsilon} < |B_1|/2$, where C and ε are the constants in Theorem 2.11. Consider the function $\tilde{u}(x) = \tau\theta^{-1}u(rx)$. It satisfies the properties

$$\begin{aligned}
&\tilde{u} \ge 0 \text{ in } B_2,\\
&\mathcal{M}^-(D^2 \tilde{u}) \le \tau\theta^{-1}r^2\varepsilon_1 \text{ in } B_2,\\
&|\{\tilde{u} \ge \tau\} \cap B_1| \ge \frac{1}{2}|B_1| > C\tau^{-\varepsilon}.
\end{aligned}$$

Let us choose $\varepsilon_1 = \tau^{-1}$. Since $\theta \ge r^2$, we have

$$\mathcal{M}^-(D^2 \tilde{u}) \le 1 \text{ in } B_2.$$

We now apply Theorem 2.11 and obtain that $\tilde{u} > 1$ in $B_{3/2}$. Scaling back, we obtain $u > \varepsilon_1\theta$ in $B_{3r/2}$. □

2.4 Hölder continuity

In this section, we derive the Hölder estimates of Theorem 2.1 from the (scaled) L^ε estimate.

Proof of Theorem 2.1. We start by normalizing the solution u. Let

$$v(x) = \frac{u(x)}{C_0(1+\varepsilon_1^{-1})},$$

where ε_1 is the constant from Lemma 2.12. The function v satisfies the estimates

$$\begin{aligned}&\mathcal{M}^-(D^2 v) \le \varepsilon_1 \text{ in } B_1,\\ &\mathcal{M}^+(D^2 v) \ge -\varepsilon_1 \text{ in } B_1,\\ &\|v\|_{L^\infty(B_1)} \le 1,\end{aligned}$$

Let $a_k = \min_{B_{2^{-k}}} v$ and $b_k = \max_{B_{2^{-k}}} v$. We will prove that for some $\alpha \in (0,1)$,

$$b_k - a_k \le 2 \times 2^{-\alpha k}. \tag{2.7}$$

For $k=0$, the statement is obvious since $b_0 \le \|v\|_{L^\infty(B_1)}$ and $a_0 \ge -\|v\|_{L^\infty(B_1)}$, thus $b_0 - a_0 \le 2$. Now we proceed by induction.

Assume that $b_k - a_k \le 2 \times 2^{-\alpha k}$ and let us prove that $b_{k+1} - a_{k+1} \le 2 \times 2^{-\alpha(k+1)}$. If $b_k - a_k \le 2 \times 2^{-\alpha(k+1)}$, then we are done since $b_{k+1} - a_{k+1} \le b_k - a_k$. Hence, we can assume that $\frac{b_k - a_k}{2} > 2^{-\alpha(k+1)}$.

Let $m_k = (a_k + b_k)/2$. We have two alternatives. Either $|\{v > m_k\} \cap B_{2^{-k-1}}| \ge |B_{2^{-k-1}}|/2$ or $|\{v \le m_k\} \cap B_{2^{-k-1}}| \ge |B_{2^{-k-1}}|/2$. In the first case we will prove that a_{k+1} is larger than a_k, whereas in the second case we will show that b_{k+1} is smaller than b_k.

Let us assume the first case, i.e. $|\{v > m_k\} \cap B_{2^{-k-1}}| \ge |B_{2^{-k-1}}|/2$. Now consider the function $v - a_k$ in $B_{2^{-k}}$. We have

$$\begin{aligned}&v - a_k \ge 0 \text{ in } B_{2^{-k}} \quad \text{since } a_k \text{ is the minimum of } v \text{ in } B_{2^{-k}},\\ &M^-(D^2(v - a_k)) \le \varepsilon_1 \text{ in } B_{2^{-k}}.\end{aligned}$$

Notice that since $\frac{b_k - a_k}{2} > 2^{-\alpha(k+1)}$, if $v(x) > m_k$ for some $x \in B_{2^{-k-1}}$, then we have

$$v(x) - a_k > m_k - a_k = \frac{b_k - a_k}{2} > 2^{-\alpha(k+1)}.$$

Thus,

$$\{v > m_k\} \subset \{v - a_k > 2^{-\alpha(k+1)}\}.$$

It follows that

$$|\{v - a_k > 2^{-\alpha(k+1)}\} \cap B_{2^{-k-1}}| \ge |\{v > m_k\} \cap B_{2^{-k-1}}| \ge |B_{2^{-k-1}}|/2.$$

We apply Lemma 2.12 to $v - a_k$ with $r = 2^{-k-1}$ and $\theta = r^\alpha > r^2$ to obtain that $v - a_k \geq \varepsilon_1 2^{-(k+1)\alpha}$ in $B_{2^{-k-1}}$ for some $\varepsilon_1 > 0$ universal. Therefore, we have that $a_{k+1} \geq a_k + \varepsilon_1 2^{-(k+1)\alpha}$. In particular

$$b_{k+1} - a_{k+1} \leq b_k - a_k - \varepsilon_1 2^{-(k+1)\alpha} \leq (2^{\alpha+1} - \varepsilon_1) 2^{-(k+1)\alpha} \leq 2 \times 2^{-(k+1)\alpha}$$

provided that α is chosen universally small enough so that $2^{\alpha+1} \leq 2 + \varepsilon_1$.

Let us assume the second case, i.e. $|\{v \leq m_k\} \cap B_{2^{-k-1}}| \geq |B_{2^{-k-1}}|/2$. This time we consider the function $b_k - v$ in $B_{2^{-k}}$. We have

$$\begin{aligned} &b_k - v \geq 0 \text{ in } B_{2^{-k}} \quad \text{since } b_k \text{ is the maximum of } v \text{ in } B_{2^{-k}}, \\ &\mathcal{M}^-(D^2(b_k - v)) \leq \varepsilon_1 \text{ in } B_{2^{-k}}. \end{aligned}$$

Notice that since $\frac{b_k - a_k}{2} > 2^{-\alpha(k+1)}$, if $v(x) \leq m_k$ for some $x \in B_{2^{-k-1}}$, then we have

$$b_k - v(x) \geq b_k - m_k = \frac{b_k - a_k}{2} > 2^{-\alpha(k+1)}.$$

Thus,

$$\{v \leq m_k\} \subset \{b_k - v > 2^{-\alpha(k+1)}\}.$$

It follows that

$$|\{b_k - v > 2^{-\alpha(k+1)}\} \cap B_{2^{-k-1}}| \geq |\{v \leq m_k\} \cap B_{2^{-k-1}}| \geq |B_{2^{-k-1}}|/2.$$

We apply Lemma 2.12 to $b_k - v$ with $r = 2^{-k-1}$ and $\theta = r^\alpha$ to obtain that $b_k - v \geq \varepsilon_1 2^{-(k+1)\alpha}$ in $B_{2^{-k-1}}$. Therefore, we have that $b_{k+1} \leq b_k - \varepsilon_1 2^{-(k+1)\alpha}$. In particular

$$b_{k+1} - a_{k+1} \leq b_k - a_k - \varepsilon_1 2^{-(k+1)\alpha} \leq (2^{\alpha+1} - \varepsilon_1) 2^{-(k+1)\alpha} \leq 2 \times 2^{-(k+1)\alpha}.$$

The estimate (2.7) implies that v is C^α at the origin, with

$$|v(x) - v(0)| \leq 4|x|^\alpha$$

for all $x \in B_1$. Scaling back to the function u, this means that for all $x \in B_1$,

$$|u(x) - u(0)| \leq 4(1 + \varepsilon_1^{-1}) C_0 |x|^\alpha \leq C C_0 |x|^\alpha,$$

where C, α are two universal positive constants. By a standard translation and covering argument, we have that $u \in C^\alpha(B_{1/2})$ and

$$[u]_{C^{0,\alpha}(B_{1/2})} \leq \tilde{C} C_0,$$

where $\tilde{C}$ differs from C by a universal constant. □

2.5 Harnack inequality

The following is the so-call weak Harnack inequality.

Theorem 2.13 (Weak Harnack inequality). *There exist two positive universal constants p_0 and C such that if u is a nonnegative lower semi-continuous function satisfying $\mathcal{M}^-(D^2u) \leq C_0$ in B_2, then*

$$\|u\|_{L^{p_0}(B_1)} \leq C(\inf_{B_{3/2}} u + C_0).$$

Proof. By replacing u with $u/(\inf_{B_{3/2}} u + C_0)$, we may assume that $C_0 = 1$ and $\inf_{B_{3/2}} u \leq 1$. By the L^ε estimate in Theorem 2.11, we have

$$|\{u \geq t\} \cap B_1| \leq Ct^{-\varepsilon}$$

for all $t > 0$. For $p_0 = \varepsilon/2$, we have

$$\int_{B_1} u^{p_0} = p_0 \int_0^\infty t^{p_0 - 1} |\{u \geq t\} \cap B_1| \mathrm{d}t \leq Cp_0 + Cp_0 \int_1^\infty t^{\varepsilon/2 - 1 - \varepsilon} \mathrm{d}t \leq C.$$

By scaling back, we have the conclusion. □

The following is the so-called local maximum principle.

Theorem 2.14 (Local maximum principle). *Suppose u is an upper semi-continuou function such that $\mathcal{M}^+(D^2u) \geq -C_0$ in B_2. Then for every $p > 0$, we have*

$$\sup_{B_{1/2}} u \leq C(p)(\|u^+\|_{L^p(B_1)} + C_0),$$

where $C(p)$ is a positive constant depending only on λ, Λ, n and p.

Proof. By replacing u by $u/(\|u^+\|_{L^p(B_1)} + C_0)$, we may assume that $C_0 \leq 1$ and $\|u^+\|_{L^p(B_1)} \leq 1$. Hence, we have

$$|\{u \geq t\} \cap B_1| \leq t^{-p} \int_{B_1} (u^+)^p \leq t^{-p} \ \ \forall\, t > 0. \tag{2.8}$$

Let $\beta \geq 1$, which will be chosen universally in the end. Define $h_s(x) = s(1 - |x|)^{-\beta}$ in B_1. We choose the minimum value of s so that $h_s \geq u$ in B_1. Consequently, we have

$$u(x) \leq h_s(x) \ \ \forall\, x \in B_1, \text{ and there exists } x_0 \in B_1 \text{ such that } u(x_0) = h_s(x_0).$$

We may also $s \geq 1$ since otherwise the conclusion follows immediately. Let

$$r = (1 - |x_0|)/2.$$

Then

$$u(x_0) = h_s(x_0) = s(2r)^{-\beta} \geq s \geq 1.$$

We are going to estimate $|\{u \geq u(x_0)/2\} \cap B_r(x_0)|$ in two different ways.

On one hand, from (2.8) we have

$$|\{u \geq u(x_0)/2\} \cap B_r(x_0)| \leq |\{u \geq u(x_0)/2\} \cap B_1| \leq 2^p(u(x_0))^{-p} = 2^p s^{-p}(2r)^{\beta p}. \quad (2.9)$$

On the other hand, let $\mu \in (0,1)$ be a small constant which will be chosen later universally. For $x \in B_{\mu r}(x_0)$, we have

$$u(x) \leq h_s(x) = s(1-|x|)^{-\beta} \leq s(2r - \mu r)^{-\beta} = u(x_0)(1-\mu/2)^{-\beta}.$$

Let

$$v(x) = \frac{u(x_0)(1-\mu/2)^{-\beta} - u(x_0 + \mu r x)}{u(x_0)(1-\mu/2)^{-\beta} - u(x_0)}.$$

Then $v \geq 0$ in B_1, $v(0) = 1$, and it satisfies

$$\mathcal{M}^-(D^2 v) \leq \frac{\mu^2 r^2}{u(x_0)(1-\mu/2)^{-\beta} - u(x_0)} \leq \frac{\mu^2}{(1-\mu/2)^{-\beta} - 1} \quad \text{in } B_1, \text{ since } u(x_0) \geq 1.$$

Now if we choose μ, β such that

$$\frac{\mu^2}{(1-\mu/2)^{-\beta} - 1} \leq 1, \quad (2.10)$$

Then we can apply Corollary 2.9 to obtain

$$|\{v \leq M\} \cap B_{1/2}| \geq \delta |B_{1/2}|,$$

where δ, M are the constants in Corollary 2.9. For $x \in B_{1/2}$ such that $v(x) \leq M$, we have

$$u(x_0 + \mu r x) \geq u(x_0)(1-\mu/2)^{-\beta} - M(u(x_0)(1-\mu/2)^{-\beta} - u(x_0)).$$

If we choose μ, β such that

$$(1-\mu/2)^{-\beta} - M((1-\mu/2)^{-\beta} - 1) \geq \frac{1}{2}, \quad (2.11)$$

then we have

$$\{v \leq M\} \cap B_{1/2} \subset \{x \in B_{1/2} : u(x_0 + \mu r x) \geq u(x_0)/2\},$$

and thus

$$|\{u \geq u(x_0)/2\} \cap B_{\mu r/2}(x_0)| \geq \delta |B_{\mu r/2}(x_0)| = \delta \omega_n (\mu r)^n 2^{-n},$$

where $\omega_n = |B_1|$. Together with (2.9), we have

$$\delta\omega_n(\mu r)^n 2^{-n} \leq 2^p s^{-p}(2r)^{\beta p}. \tag{2.12}$$

Now we let $\beta = \max(np^{-1}, 1)$ and we can choose μ sufficiently small such that (2.10) and (2.11) hold. Consequently, (2.12) holds for $\beta = \max(np^{-1}, 1)$, from which we obtain a universal upper bound of s. This proves the conclusion. □

Proof of Theorem 2.2: The Harnack inequality in Theorem 2.2 follows from Theorem 2.13 and Theorem 2.14. □

3 $C^{1,\alpha}$ estimates

The following proposition follows from Jensen approximation.

Proposition 3.1. *Let u be a solution of $F(D^2u) = 0$ in B_1. Let $h > 0$ and $e \in \mathbb{R}^n$ with $|e| = 1$. Then for $v_h(x) = u(x + he) - u(x)$, it satisfies*

$$\mathcal{M}^-(\frac{\lambda}{n}, \Lambda, D^2 v_h) \leq 0 \leq \mathcal{M}^+(\frac{\lambda}{n}, \Lambda, D^2 v_h) \quad \text{in } B_{1-h}.$$

Using the difference quotient method, we can obtain $C^{1,\alpha}$ estimate.

Theorem 3.2. *Let u be a solution of $F(D^2u) = 0$ in B_1. Then $u \in C^{1,\alpha}(B_{1/2})$, and there holds*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + |F(0)|),$$

where $\alpha \in (0, 1)$ and $C > 0$ are universal constants.

Sketch of proof: First of all, we know from Theorem 2.1 that

$$u \in C^\alpha(B_1).$$

Let

$$v_\beta = \frac{v_h}{h^\beta}.$$

By Proposition 3.1 and Theorem 2.1 we have that

$$v_\alpha \in C^\alpha(B_{1-h}),$$

and consequently,

$$u \in C^{2\alpha}(B_{1-h}).$$

By applying Proposition 3.1 and Theorem 2.1 to $v_{2\alpha}$, we have $v_{2\alpha} \in C^\alpha$, and thus $u \in C^{3\alpha}(B_{1-h})$. We can repeat this process to obtain that $u \in C^{0,1}$. Finally, we apply Proposition 3.1 and Theorem 2.1 to v_1 to obtain that $v_1 \in C^\alpha$, and thus $u \in C^{1,\alpha}$. □

Now one may wonder whether solutions of $F(D^2u) = 0$ in general are of $C^{1,1}$. The answer is

- yes, when the dimension $n = 2$. The solutions will be $C^{2,\alpha}$ indeed. The reason is that in 2-D, the linear equation in nondivergence form can be written in divergence form. This is a very classical results, which may be due to Nirenberg in early 60's.
- still unknown, when the dimension $n = 3$ or 4.
- no, when the dimension $n \geq 5$ even with F smooth. The counterexamples are found by Nadirashvili-Vlăduţ in [9] which shows that $C^{1,\alpha}$ is optimal. When $n = 9$, a counterexample for not C^2 solution can be

$$u(x) = |x|^{-1} \cdot \det \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix},$$

which will be a solution of some uniformly elliptic equation, but is clearly not C^2.

Note that all the known counterexamples have only one isolated singularity at the origin. One may also ask how big the size of the set of singular points can be. It was shown in [1] that if we assume F is C^1 (instead of Lipschitz) then u is of $C^{2,\alpha}$ for every $\alpha \in (0,1)$ outside a closed set of Hausdroff dimension at most $n - \varepsilon$ for some (small) universal positive constant ε.

However, if we put extra assumption, such as convexity, on F, then it was shown by Evans and Krylov independently that the solutions will be $C^{2,\alpha}$. This is the topic in the next section.

4 The Evans-Krylov theorem

Theorem 4.1. *Let F be uniformly elliptic and concave, and let u be a viscosity solution of $F(D^2u) = 0$ in B_1. Then $u \in C^{2,\alpha}(B_{1/2})$ and there holds*

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + |F(0)|),$$

where $C > 0$ and $\alpha \in (0,1)$ are universal constant.

In this particular proof presented here as in [4, 5], we will assume that the solutions are classical and prove the a priori estimates. There is no loss of generality of doing this, since we can later solve a corresponding Dirichlet problem as long as we have a priori estimate, and the conclusion follows from the uniqueness of the viscosity solution. Also, one can assume that $F(0) = 0$.

The proof of this theorem has two clearly divided parts. These are

$$\|u\|_{C^{1,1}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)} \tag{4.1}$$

and

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C\|u\|_{C^{1,1}(B_1)}. \tag{4.2}$$

We will see the proofs of these two estimates separately.

4.1 The assumptions revisited

We said that F is uniformly elliptic. This means by definition that

$$\mathcal{M}^-(X-Y) \leq F(X) - F(Y) \leq \mathcal{M}^+(X-Y).$$

We know that at every point $D^2u(x)$ is a symmetric matrix for which $F(D^2u(x)) = 0$. Thus

$$\begin{aligned}\mathcal{M}^-(D^2u(x) - D^2u(y)) \leq F(D^2u(x)) - F(D^2u(y)) = 0,\\ \mathcal{M}^+(D^2u(x) - D^2u(y)) \geq F(D^2u(x)) - F(D^2u(y)) = 0.\end{aligned}$$

These two relations mean that the sum of the positive and negative eigenvalues of $(D^2u(x) - D^2u(y))$ are comparable. More precisely,

$$\frac{\lambda}{\Lambda} tr(D^2u(x) - D^2u(y))^- \leq tr(D^2u(x) - D^2u(y))^+ \leq \frac{\Lambda}{\lambda} tr(D^2u(x) - D^2u(y))^- \tag{4.3}$$

We could rephrase the above as that $\|(D^2u(x) - D^2u(y))^-\| \approx \|(D^2u(x) - D^2u(y))^+\|$, since for positive definite matrices the trace and norm are comparable.

The concavity and translation invariance of F makes the second order incremental quotients be subsolutions of an equation. More precisely, since F is concave we have that

$$F\left(\frac{D^2u(x+h) + D^2u(x-h)}{2}\right) \geq \frac{F(D^2u(x+h)) + F(D^2u(x-h))}{2} = 0.$$

Therefore

$$\begin{aligned}&\mathcal{M}^+\left(\frac{D^2u(x+h) + D^2u(x-h) - 2D^2u(x)}{2}\right)\\ &\geq F\left(\frac{D^2u(x+h) + D^2u(x-h)}{2}\right) - F(D^2u(x))\\ &= 0.\end{aligned}$$

Because of the homogeneity of $\mathcal{M}^+$, $\mathcal{M}^+(D^2v) \geq 0$ where $v(x)$ is a second order incremental quotient

$$v(x) = \frac{u(x+h) + u(x-h) - 2u(x)}{|h|^2}.$$

Passing to the limit (if u is regular enough) we see that for a second derivative in any direction e we have

$$\mathcal{M}^+(D^2\partial_{ee}u) \geq 0.$$

The computation above can be repeated to obtain

$$\mathcal{M}^+(D^2(a_{ij}\partial_{ij}u)) \geq 0, \tag{4.4}$$

for any positive semidefinite matrix a_{ij}. This is because $a_{ij}\partial_{ij}u(x)$ can be approximated with second order incremental quotients which are a weighted average of values of u in points near x minus the value of u at x.

The two formulas (4.3) and (4.4) are the basis of the proof of Evans-Krylov theorem.

4.2 The $C^{1,1}$ a priori estimate

The idea of the $C^{1,1}$ a priori estimate is simple to say. From (4.4), we know that the second derivatives are subsolutions to an equation. This will imply that they are all bounded above. Then, from the equation $F(D^2u(x)) = 0$ and the uniform ellipticity of F, we immediately conclude that they are bounded below as well.

Step 1: $\Delta u \in L^1$.

Recall that F is concave on the space $\mathcal{S}$ of symmetric matrices. Thus there is a support hyperplane from above at $0 \in \mathcal{S}$. Suppose that this hyperplane is $L(M) = tr(AM)$, since $F(0) = 0$. By change of variables, we may assume that $A = I$. Then, we have

$$\Delta u(x) = L(D^2u(x)) \geq F(D^2u(x)) = 0$$

Let $b : B_1 \to \mathbb{R}$ be a nonnegative smooth function compactly supported inside B_1 such that $b \equiv 1$ in $B_{1/2}$, then

$$\|\Delta u\|_{L^1(B_{1/2})} \leq \int_{B_1} b(x)\Delta u(x)dx = \int_{B_1} \Delta b(x)u(x)dx \leq C\|u\|_{L^\infty(B_1)},$$

where $C > 0$ is a universal constant.

Step 2: $u \in W^{2,2}$.

From (4.4), $\mathcal{M}^+(D^2\Delta u) \geq 0$, $\Delta u \geq 0$ in B_1, and $\Delta u \in L^1_{loc}(B_1)$. It follows from *Step 1* and the local maximum principle in Theorem 2.14 for $p = 1$ that

$$0 \leq \Delta u \leq C \quad \text{in } B_{3/4},$$

where $C > 0$ is a universal constant. Using Calderon-Zygmund theory, then $D^2u(x) \in L^p(B_{1/2})$ for any $p < \infty$. For the remainder of the proof we will only use the simplest estimate $D^2u(x) \in L^2(B_{1/2})$.

Step 3: $u \in C^{1,1}$.

Since we got that every second derivative $\partial_{ee}u$ is in $L^2_{loc}(B_1)$ and satisfies (4.4), then by the local maximum principle in Theorem 2.14, they are all bounded above.

We now use the equation again. From the uniform ellipticity we have that

$$M^-(D^2u(x)) \leq F(D^2u(x)) - F(0) \leq M^+(D^2u(x)).$$

This means that if the positive part $\|D^2u(x)^+\|$ is bounded, then the negative part $\|D^2u(x)^-\|$ must be bounded as well.

This concludes the proof of the $C^{1,1}$ estimate in (4.1).

4.3 The $C^{2,\alpha}$ a priori estimate

We want to set up an improvement of oscillation iteration for the Hessian D^2u. More precisely, we want to show that for some universal constants $C > 0$ and $\theta > 0$ such that

$$\sup_{B_{2^{-k}}} \|D^2u(x) - D^2u(0)\| \leq C\|u\|_{C^{1,1}}(1-\theta)^k.$$

There is nothing special about the origin, so if we prove the above inequality, we prove the interior estimate by translating the result. We define P and N to be the sum of positive and negative eigenvalues of $D^2u(x) - D^2u(0)$.

$$P(x) = tr(D^2u(x) - D^2u(0))^+,$$
$$N(x) = tr(D^2u(x) - D^2u(0))^-.$$

We are using the convention that $A = A^+ - A^-$, so both P and N are non negative quantities. Because of (4.3), the quantities $P(x)$, $N(x)$ and $\|D^2u(x) - D^2u(0)\|$ are all comparable (meaning that the ratio between any two is bounded below and above)

$$P(x) \approx N(x) \approx \|D^2u(x) - D^2u(0)\|.$$

Since all these quantities are comparable, we only have to prove that

$$\sup_{B_{2^{-k}}} P(x) \leq C\|u\|_{C^{1,1}}(1-\theta)^k. \tag{4.5}$$

We will proceed with an iterative improvement of oscillation, as in the proofs of Holder continuity. This time, instead of the oscillation, we are decreasing the maximum of P in $B_{2^{-k}}$ in each step.

By the standard scaling of the equation, we reduce the problem to unit scale. We must show that if $P \leq 1$ in B_1, then $P \leq (1-\theta)$ in $B_{1/2}$ for some $\theta > 0$.

The following simple observation is useful. One way to characterize P is

$$P(x) = \max_{\{a_{ij}\} \text{ orthogonal proj. matrix}} a_{ij}(\partial_{ij}u(x) - \partial_{ij}u(0)).$$

Indeed, suppose v_k is one normalized eigenvector for the eigenvalue λ_k of a matrix U. Then

$$tr(v_k v_k^T U) = \lambda_k.$$

Thus the orthogonal projection matrix to realize P is the matrix consisting the normalized eigenvectors for the positive eigenvalues of $D^2u(x) - D^2u(0)$.

Let x_0 be the point in $\overline{B_{1/2}}$ such that $P(x_0) = \max_{B_{1/2}} P(x_0)$. Let a_{ij} be the orthogonal projection matrix to the eigenspace of $D^2u(x_0) - D^2u(0)$ corresponding to positive eigenvalues. Thus, we have

$$P(x_0) = a_{ij}(\partial_{ij}u(x_0) - \partial_{ij}u(0)).$$

Let

$$v(x) = a_{ij}(\partial_{ij}u(x) - \partial_{ij}u(0)).$$

We have $v(x_0) = P(x_0)$ and $v(x) \leq P(x) \leq 1$ in B_1. We are going to show that

$$v(x_0) \leq 1 - \theta.$$

We argue by contradiction that

$$v(x_0) > 1 - \theta. \tag{4.6}$$

By (4.4), we know that $\mathcal{M}^+(D^2v) \geq 0$ in B_1. Then we have that

$$\begin{aligned} 1 - v &\geq 0 \quad \text{in } B_1 \\ \mathcal{M}^-(D^2(1-v)) &\leq 0 \quad \text{in } B_1. \end{aligned}$$

By the L^ε estimate in Theorem 2.11 that (choosing $t = \theta^{-1/2}$)

$$|\{(1-v) \geq \theta^{-\frac{1}{2}} \inf_{B_{1/2}} (1-v)\} \cap B_{1/4}| \leq C\theta^{\varepsilon/2}.$$

By the contradiction hypothesis (4.6), we have $\inf_{B_{1/2}}(1-v) \leq \theta$. Then we have

$$|\{(1-v) \geq \theta^{\frac{1}{2}}\} \cap B_{1/4}| \leq C\theta^{\varepsilon/2}. \tag{4.7}$$

Let $\Omega = \{(1-v) < \theta^{\frac{1}{2}}\} \cap B_{1/4}$. Then for $x \in \Omega$, we have

$$1 - \theta^{1/2} < v(x) \leq P(x) \leq 1. \tag{4.8}$$

Therefore, for $x \in \Omega$,

$$P(x) - v(x) \le \theta^{1/2}. \tag{4.9}$$

Let

$$b_{ij} = I - a_{ij}$$

and

$$w(x) = b_{ij}(\partial_{ij}u(x) - \partial_{ij}u(0)).$$

Since

$$\Delta u = w + v = P - N,$$

we have for $x \in \Omega$

$$\begin{aligned} w(x) &= P(x) - v(x) - N(x) \\ &\le \theta^{1/2} - N(x) \\ &\le \theta^{1/2} - \frac{\lambda}{\Lambda}P(x) \\ &\le (1 + \lambda/\Lambda)\theta^{1/2} - \lambda/\Lambda \le -\lambda/(2\Lambda) := -c_0 \qquad \text{for } \theta \text{ small enough,} \end{aligned}$$

where we used (4.9), (4.3) and (4.8). Since $\mathcal{M}^+(D^2(w + c_0)) \ge 0$ in B_1, it follows from the local maximum principle in Theorem 2.14 that

$$\sup_{B_{1/8}}(w + c_0) \le C\|(w + c_0)^+\|_{L^1(B_{1/4})} = C\|(w + c_0)^+\|_{L^1(B_{1/4}\setminus\Omega)}.$$

Since $P \le 1$ in B_1, then $N \le \Lambda/\lambda$ in B_1. Hence, $w \le C$ in B_1. Therefore,

$$c_0 = w(0) + c_0 \le \sup_{B_{1/8}}(w + c_0) \le C|B_{1/4} \setminus \Omega| \le C\theta^{\varepsilon/2}.$$

This is a contradiction if θ is small. Thus we have shown that if

$$P \le 1 \text{ in } B_1,$$

then

$$P \le (1 - \theta) \text{ in } B_{1/2} \text{ for some } \theta > 0.$$

By scaling, we may consider

$$P_k(x) = (1 - \theta)^{-k} P(2^{-k}x)$$

to obtain that

$$P_k(x) \le 1 - \theta \text{ in } B_{1/2}$$

from which (4.5) follows. This implies that

$$P(x) \le C|x|^\alpha \quad \text{for } x \in B_1,$$

which is equivalent to

$$\|D^2u(x) - D^2u(0)\| \le C|x|^\alpha.$$

This finishes the proof of (4.2) as well as Theorem 4.1.

5 $W^{2,p}$ estimates

Let Ω be a bounded domain in $\mathbb{R}^n$ and u is a continuous function in Ω. We define, any every open set $H \subset \Omega$ and $M \in (0, \infty]$ that

$$\begin{aligned} &\underline{G}_M(u, H) \\ &= \underline{G}_M(H) \\ &= \{x_0 \in H : \text{there exists a concave paraboloid of opening } M \text{ touching } u \text{ from below in } H\}. \end{aligned}$$

Let $\underline{A}_M(H) = H \setminus \underline{G}_M(H), \overline{G}_M(u, H) = \underline{G}_M(-u, H), \overline{G}_M(H) = \overline{G}_M(u, H), \overline{A}_M(H) = H \setminus \overline{G}_M(H), G_M(H) = \overline{G}_M(H) \cap \underline{G}_M(H), A_M(H) = H \setminus G_M(H)$.

Lemma 5.1. *Let $B_5 \subset \Omega$, $\mathcal{M}^-(D^2u) \le f$ in Ω, $u \ge 0$ in Ω, $\inf_{B_{1/4}} u \le 1$, $\|f\|_{L^n(B_5)} \le \varepsilon_0$. Then*

$$\underline{G}_M(u, \Omega) \cap B_1 \ge \delta,$$

where δ, ε_0, M are universal constants.

Proof. The proof is the same as the proof of Proposition 2.4. We repeat it here for completeness.

Let $x_0 \in B_{1/4}$ be such that $u(x_0) \le 1$. For every $x \in B_{1/4}$, let $y \in \overline{B}_5$ be a point where the minimum of $u(z) + 4|z - x|^2$, which is a function of z, is achieved. This is the same as that we slide the parabola $-4|z - x|^2$ from the below of u until they touch and y is a touch point. Note that

- When $z \in \Omega \setminus B_1$, then $u(z) + 4|z - x|^2 \ge 0 + 4|1 - 1/4|^2 = 9/4$.
- $u(x_0) + 4|x_0 - x|^2 \le 1 + 4|1/4 + 1/4|^2 = 2 < 9/4$.

Therefore, for every $x \in B_{1/4}$, such minimum point $y \in B_1$, and $u(y) + 4|y - x|^2 \le u(x_0) + 4|x_0 - x|^2 \le 2$. In particular, $u(y) \le 2$. Note that for one value of x, there could be more than one point y where the minimum is achieved. However, the value of y uniquely determines x, since we must have

$$\nabla u(y) + 8(y - x) = 0, \quad D^2u(y) + 8I \ge 0. \tag{5.1}$$

Thus,

$$x = y + \frac{\nabla u(y)}{8}.$$

We define this as a map $x = m(y) = y + \frac{\nabla u(y)}{8}$, which is onto $B_{1/4}$. Consequently, we have that

$$\nabla m(y) = I + \frac{D^2 u(y)}{8}$$

Since for each y we know $u(y) \leq 2$, the domain U of the map m satisfies that $U \subset \{y \in B_1 : u(y) \leq 2\}$. Thus, we have

$$|B_{1/4}| \leq \int_U |\det \nabla m(y)| \mathrm{d}y \leq \int_U |\det(I + \frac{D^2 u(y)}{8})|$$

On the other hand, from the inequality $D^2 u(y) + 8I \geq 0$ in (2.3) and the equation $\mathcal{M}^-(D^2 u) \leq f$ it follows that

$$|D^2 u| \leq C(1 + |f|)$$

for some positive constant C depending only on λ, Λ, n. Thus, we have

$$|B_{1/4}| \leq C|U| + C\|f\|_{L^n(B_1)}.$$

We can choose ε_0 universally small, so that $|\underline{G}_8(u, \Omega) \cap B_1| \geq |U| \geq \delta$. □

Lemma 5.2. *Let $B_5 \subset \Omega$, $\mathcal{M}^-(D^2 u) \leq f$ in Ω, $\underline{G}_1(u, \Omega) \cap B_1 \neq \emptyset$, $\|f\|_{L^n(B_5)} \leq \varepsilon_0$. Then*

$$|\underline{G}_M(u, \Omega) \cap B_1| \geq \delta$$

where ε_0, δ, M are universal constants.

Proof. This corresponds to Lemma 2.7. Let P be the polynomial of opening 1 touching u at a point $x_0 \in B_1$ and $u \leq P$ in Ω. Let $v = u - P$. Then $v \geq 0$ in Ω, $v(x_0) = 0$ and

$$\mathcal{M}^-(D^2 v) \leq f + 1 \quad \text{in } \Omega.$$

We claim that $\inf_{B_{1/8}} v(z) \leq M_0$ for M_0 universally large.

We argue by contradiction. If not, then $v \geq M_0$ everywhere in $B_{1/4}$. Let

$$\varphi_0(x) = \frac{|x|^{-p} - 2^{-p}}{8^p - 2^{-p}} \quad \text{in } \mathbb{R}^n \setminus B_{1/8}, \quad \text{and} \quad \varphi_0 \equiv 1 \quad \text{in } B_{1/8}.$$

For $\varphi = C_0 \varphi_0$, we can choose C_0 and p universally large so that $\varphi \geq 2$ in $B_{3/2}$ and

$$\mathcal{M}^-(D^2 \varphi) \geq 2 \quad \text{in } B_4 \setminus \overline{B}_{1/8}.$$

For every $x \in B_{1/8}$, slide $\varphi(z-x)$ from below until it touches v. If we choose M_0 large, the touch point(s) y has to be in $B_3 \setminus B_{1/8}(x)$. Then we have

$$\nabla v(y) = \nabla \varphi(y-x), D^2 v(y) \geq D^2 \varphi(y-x).$$

Note that for one value of x, there could be more than one point y where the minimum is achieved. However, the value of y uniquely determines x:

$$\nabla v(y) = \nabla \varphi(y-x) = C(C_0,p)|y-x|^{-p-2}(y-x), \quad |\nabla v(y)| = C(C_0,p)|y-x|^{-p-1},$$

and therefore, $|y-x| = \left(\frac{C(C_0,p)}{|\nabla v(y)|}\right)^{\frac{1}{p+1}}$,

$$x = y - \frac{\nabla v(y)}{C(C_0,p)} \left(\frac{C(C_0,p)}{|\nabla v(y)|}\right)^{\frac{p+2}{p+1}} =: m(y).$$

This map $y \to m(y)$ is onto $B_{1/8}$. Let U be the domain of m, i.e., U consists of the touching points. Then, we have for $y \in U$ that

$$f(y) + 1 \geq \mathcal{M}^-(D^2 v)(y) \geq \mathcal{M}^-(D^2 \varphi)(y-x) \geq 2.$$

This means that $f \geq 1$ at those touching points.

Meanwhile, by the equation and $D^2 v(y) \geq D^2 \varphi(y-x)$, we have

$$|D^2 v(y)| \leq C(|f(y)| + 1 + |D^2 \varphi|(y - m(y))) \quad y \in U.$$

Moreover, we have

$$D^2 v(y) = D^2 \varphi(y-x)(I - \nabla m(y)),$$

that is,

$$\nabla m(y) = I - (D^2 \varphi(y - m(y)))^{-1} D^2 v(y).$$

Then we have again

$$|B_{1/8}| \leq C \int_U |\det \nabla m(y)| \leq C \int_U \left(1 + \frac{|f| + 1 + |D^2 \varphi|}{|D^2 \varphi|}\right)^n \leq C \int_U |f|^n \leq C\varepsilon_0.$$

This is a contradiction.

Now let $w = v/M_0 \geq 0$. We have $\mathcal{M}^-(D^2 w) \leq \frac{f+1}{M_0}$ in Ω and $\inf_{B_{1/4}} w \leq 1$. We can assume M_0 large so that $1/M_0 \leq \varepsilon_0$. Then all the assumptions in Lemma 5.1 are satisfied for w. So we have

$$|\underline{G}_8(w, \Omega) \cap B_1| \geq \delta.$$

This implies that $|\underline{G}_{8M_0+1}(u, \Omega) \cap B_1| \geq \delta$. So we can choose $M = 8M_0 + 1$. □

Proposition 5.3. *Under the assumptions of Lemma 5.1, we have*

$$|\underline{A}_t(u,\Omega) \cap B_1| \leq Ct^{-\mu},$$

where C, μ *are positive universal constants.*

Proof. Let

$$\begin{aligned} A_k &= \underline{A}_{M^k}(u,\Omega) \cap B_1, \\ D_k &= (\underline{A}_{M^k}(u,\Omega) \cap B_1) \cup \{x \in B_1 : m(f^n) \geq (cM^k)^n\}, \\ a_k &= |A_k|,\ d_k = |D_k|,\ b_k = |\{x \in B_1 : m(f^n) \geq (cM^k)^n\}|. \end{aligned}$$

where $c = \varepsilon_0/6^n$ and $m(g)(x) = \sup_{r>0} \frac{1}{B_r(x)} \int_{B_r(x)} |g(y)|dy$.

We know from Lemma 5.1 that $a_k \leq (1-\delta)|B_1|$ for all $k \in \mathbb{N}$.

For any ball $B = B_r(x_0) \subset B_1$, if $|\underline{A}_{M^{k+1}}(u,\Omega) \cap B| > (1-\delta)|B|$, then we must have $B \subset D_k$. Otherwise, there exists $z_0 \in B_r(x_0)$ such that $z_0 \in \underline{G}_{M^k}(u,\Omega)$ and

$$\sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f|^n \leq (cM^k)^n.$$

Rescaling

$$\tilde{u}(x) = \frac{1}{M^k r^2} u(x_0 + rx), \quad \tilde{\Omega} = \{(y - x_0)/r : y \in \Omega\}.$$

Then $\mathcal{M}^-(D^2\tilde{u}) \leq \frac{f(x_0+rx)}{M^k} =: \tilde{f}(x)$ in $\tilde{\Omega}$, and $\underline{G}_1(\tilde{u},\tilde{\Omega}) \cap B_1 \neq \emptyset$. Note that $B_5 \subset \tilde{\Omega}$. Moreover,

$$\begin{aligned} \int_{B_5} |\tilde{f}(y)|^n dy &= \frac{1}{M^{nk}} \int_{B_5} |f(x_0 + rx)|^n dx \\ &= \frac{1}{r^n M^{nk}} \int_{B_{5r}} |f(x_0 + x)|^n dx \\ &\leq \frac{1}{r^n M^{nk}} \int_{B_{6r}(z_0)} |f(x)|^n dx \\ &\leq 6^n c \\ &= \varepsilon_0 \end{aligned}$$

if we choose $c = \varepsilon_0/6^n$. Therefore, we can apply Lemma 5.2 that

$$|\underline{G}_M(\tilde{u},\tilde{\Omega}) \cap B_1| \geq \delta|B_1|.$$

This is

$$|\underline{G}_{M^{k+1}}(u,\Omega) \cap B_1| \geq \delta|B_1|,$$

which contradicts with $|\underline{A}_{M^{k+1}}(u,\Omega)\cap B|>(1-\delta)|B|$.

Applying the growing ink-spots lemma, Lemma 2.10, we have that

$$a_{k+1}\le(1-c\delta)d_k.$$

This implies that

$$a_k\le(1-c\delta)^k+\sum_{i=0}^{k-1}(1-c\delta)^{k-i}b_i.$$

Since the maximal function is weak type (1,1), we have that $b_k\le C\|f^n\|_{L^1}(cM^k)^{-n}=CM^{-kn}$. Let $\sigma=\max(1-c\delta,M^{-n})$. Then we have

$$a_k\le\sigma^k+\sum_{i=0}^{k-1}\sigma^{k-i}C\sigma^i=(1+Ck)\sigma^k\le C\tilde{\sigma}^k$$

if we choose $\tilde{\sigma}=\frac{1+\sigma}{2}$. □

Theorem 5.4. *There exists* $\varepsilon>0$ *such that if* $\mathcal{M}^-(D^2u)\le f,\mathcal{M}^+(D^2u)\ge f$ *in* B_1 *with* $f\in L^n(B_1)$ *then* $u\in W^{2,\varepsilon}(B_{1/2})$*, and*

$$\|u\|_{W^{2,\varepsilon}(B_{1/2})}\le C(\|u\|_{L^\infty(B_1)}+\|f\|_{L^n(B_1)}).$$

Here the constant C,ε *are universal.*

Proof. Consider $\tilde{u}=u-\inf_{B_1}u$ instead. It follows from Proposition 5.3 that

$$|\underline{A}_t(u,\Omega)\cap B_1|\le Ct^{-\mu},\quad|\overline{A}_t(u,\Omega)\cap B_1|\le Ct^{-\mu}.$$

Therefore,

$$|A_t(u,\Omega)\cap B_1|\le Ct^{-\mu}.$$

This finishes the proof. □

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