# POINCARÉ-BENDIXSON'S THEOREM: APPLICATIONS AND THE PROOF

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This lecture note is written for my two lectures of MATH 4999 (Pure Math track). The content is extracted from a chapter of lecture notes written by the author for a course on Theory of ODEs at Brown University in Fall '13 and '14. Minor revision was made so that it becomes self-contained and suits this capstone course. Students who are interested in other parts of the lecture notes (mainly about existence, uniqueness and stability) may download the full set from the Brown course website:

`https://sites.google.com/a/brown.edu/math111-fall-14`

## 1. Motivation

Ordinary Differential Equations (ODEs) are equations whose unknowns functions have only one independent variable, which will be always denoted by $t$ in this note. A system of ODEs is a number of simultaneous ODEs with one or more unknown functions (e.g. $x(t)$, $y(t)$). Here is an example of a first-order system of ODEs:

$$\begin{aligned}\frac{dx}{dt} &= -x + ay + x^2 y\\ \frac{dy}{dt} &= b - ay - x^2 y.\end{aligned}$$

The variable $t$, often regarded as the *time*, is the independent variable. The functions $x(t)$ and $y(t)$ are the unknowns of the system, whereas $a$ and $b$ are constants.

This system governs the *glycolysis* inside a human body. Here $x$ is the concentration of ADP (adenosine diphosphate) and $y$ is the concentration of F6P (fructose 6-phosphate). The rate of change of each of the two chemicals are governed by the above ODE system. For instance, one can see that the increase of $y$ will lead to slower growth rate of $y$ and higher growth rate of $x$.

Given an initial condition $x(0) = x_0$ and $y(0) = y_0$, if one can *solve* the above ODE system then we are able to predict the concentration of these two chemicals in the future. However, even for a simple-looking system like this one, an *explicit* solution is very difficult to find!

Generally speaking, mathematicians had abandoned the search for explicit solutions of this kind of non-linear systems, and instead study the *qualitative* features of the system such as existence and uniqueness, stability and periodicity. In fact, life scientists may not really care about the exact solution of the above glycolysis system, but they may be concerned more on whether the system can maintain a *sustainable* metabolism. In mathematical terms, scientists want to know whether the concentrations of these two chemicals exhibit a *periodic* pattern, so that even after some perturbation on the concentrations, they can be back to the original state (i.e. being *normal* again) after some time.

In this lecture notes, we will introduce and give the proof of a celebrated result – Poincaré-Bendixson's Theorem – which can show a *periodic solution* exists for the glycolysis system.

## 2. Phase Portrait

The general form of a first-order system of coupled ODEs is:

$$x' = P(x,y)$$
$$y' = Q(x,y)$$

We often put a system of ODEs in a **vector form**. One key reason of doing that is to allow *geometry* kicks in! To represent an ODE system in vector form, we let $\mathbf{x}(t)$ be the **unknown vector** whose components are the unknown functions $x(t)$ and $y(t)$, i.e.

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

We also put the right-hand side of an ODE system into a vector by letting:

$$\mathbf{F}(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}.$$

Since the vector $(x,y)$ can be represented by $\mathbf{x}$, we can further abbreviate $\mathbf{F}(x,y)$ by $\mathbf{F}(\mathbf{x})$. Therefore, a general system of ODEs can be written in the form of:

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}).$$

The vector form of an ODE system links the theory of ODEs with geometry. We think of a solution $\mathbf{x}(t)$ as a **parametrized curve**, $\mathbf{x}(t) = (x(t), y(t))$ in $\mathbb{R}^2$. Therefore, we will often call a solution $\mathbf{x}(t)$ as a **solution curve**, a **trajectory** or an **orbit**. The $t$-derivative, $\mathbf{x}'(t) = (x'(t), y'(t))$, represents the **tangent**, or **velocity**, vector of the curve at time $t$.

The right-hand side of the equation, written as $\mathbf{F}(\mathbf{x})$, defines a **vector field** on $\mathbb{R}^2$. The vector field corresponding to the glycolysis system is given by:

$$\mathbf{F}(x,y) = \begin{bmatrix} -x + ay + x^2 y \\ b - ay - x^2 y \end{bmatrix}$$

and its vector field plot of the special case $(a,b) = (\frac{1}{10}, \frac{1}{2})$ is shown in Figure 1 .

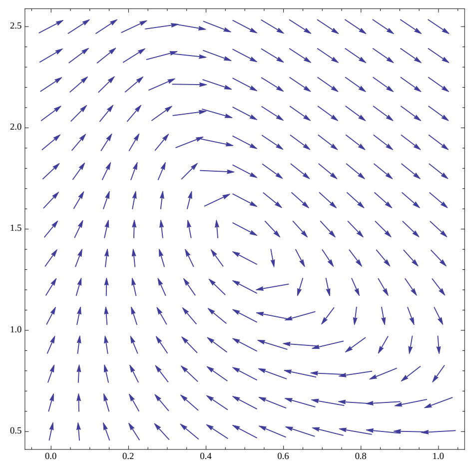


Figure 1. The vector field plot of the glycolysis system

In order for $\mathbf{x}(t)$ to be a solution curve, it has to satisfy $\mathbf{x}' = \mathbf{F}(\mathbf{x})$. Geometrically, it means that the tangent vector $\mathbf{x}'(t)$ of the curve is at any time equal to the vector field $\mathbf{F}$ at the point $\mathbf{x}(t)$. To put it in an even simpler terms, the solution curve $\mathbf{x}(t)$ *flows* along the vector field $\mathbf{F}$ at any time. Figure 2 shows the relation between the family of solution curves (in red) and the vector field (in blue) of the glycolysis example.

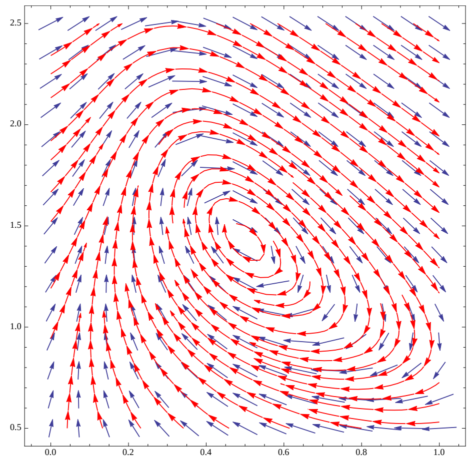


FIGURE 2. The vector field plot with solution curves with of the glycolysis system

A plot consisting of only solution curves of an ODE system is called the **phase portrait** of the system.

**Definition 2.1** (Periodic Solutions)**.** Given an ODE system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$, a solution $\mathbf{x}(t)$ of the system is said to be a **periodic solution**, or a **closed orbit**, if there exists a time $T > 0$ such that $\mathbf{x}(t+T) = \mathbf{x}(t)$ for any $t \in \mathbb{R}$. $\square$

The trajectory for a periodic solution is a **closed loop** on the phase portrait. Note that an equilibrium solution (i.e. a constant solution) is *a fortiori* periodic since $T$ can be taken to be any positive number. A periodic solution which is not a constant is called a **non-trivial periodic solution**.

## 3. FLOW MAP

Consider a system of ODE $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ on $\mathbb{R}^2$ with a $C^1$ vector field $\mathbf{F}$. Given any initial data $\mathbf{x}(0) = \mathbf{x}_0$, the Existence and Uniqueness Theorem guarantees there always exists a unique solution $\mathbf{x}(t)$. From now on, we will denote this solution by:

$$\varphi_t(\mathbf{x}_0)$$

meaning that the point $\mathbf{x}_0$ flows along the given vector field $\mathbf{F}$ for $t$ unit time.

Essentially, $\varphi_t(\mathbf{x}_0)$ is exactly the same as $\mathbf{x}(t)$ which solves $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ with initial condition $\mathbf{x}(0) = \mathbf{x}_0$. There is many advantages of using the $\varphi_t$-notation. Suppose we have another initial condition, say $\mathbf{y}_0$, we may simply denote the solution by $\varphi_t(\mathbf{y}_0)$ instead of making up a new symbol $\mathbf{y}(t)$ for that solution. Even more importantly, the $\varphi_t$ satisfies:

- For any real number $t$ and $s$, and $\mathbf{x}_0 \in \mathbb{R}^2$, we have $\varphi_t(\varphi_s(\mathbf{x}_0)) = \varphi_{t+s}(\mathbf{x}_0)$.
- For each fixed $t$, the map $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous map.

A trajectory starting from $\mathbf{x}_0$ is given by $\varphi_t(\mathbf{x}_0)$ (regarding $t$ as the independent variable). In order for $\varphi_t(\mathbf{x}_0)$ to be periodic, it suffices to have a time $T > 0$ such that $\varphi_T(\mathbf{x}_0) = \mathbf{x}_0$, since it automatically implies $\varphi_{t+T}(\mathbf{x}_0) = \varphi_t(\mathbf{x}_0)$ for any $t \in \mathbb{R}$ using the fact that $\varphi_T \circ \varphi_t = \varphi_{t+T}$.

The phase portrait of the glycolysis system (with $a = \frac{1}{10}$ and $b = \frac{1}{2}$) is shown in Figure 3. The phase portrait suggests that there should be a periodic solution (closed orbit). However, due to some unavoidable numerical errors of the plotting software, the periodic solution cannot be clearly shown in the diagram. We will show that such a periodic solution indeed *exists* using the Poincaré-Bendixson's Theorem in the next section.

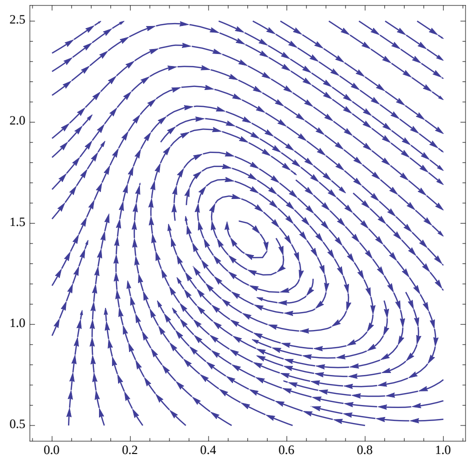


FIGURE 3. The phase portrait of the system:

## 4. POINCARÉ-BENDIXSON'S THEOREM: APPLICATIONS

In this section, we will state an important result in qualitative theory of ODEs, the Poincaré-Bendixson's Theorem, and use it to prove that a periodic solution really exists in glycolysis system. While the theorem cannot tell what is the explicit expression of the periodic solution, it gives us an idea of where the closed orbit is located in the phase portrait.

**Theorem 4.1** (Poincaré-Bendixson's Theorem)**.** *Let $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be a $C^1$ vector field in $\mathbb{R}^2$ and consider the system $\boldsymbol{x}' = \boldsymbol{F}(\boldsymbol{x})$. Suppose $K$ is a set in $\mathbb{R}^2$ such that:*

*(1) $K$ is closed and bounded;*
*(2) the system has no equilibrium point in $K$; and*
*(3) $K$ contains a forward trajectory of the system, i.e. there exists $\boldsymbol{x}_0 \in K$ such that $\varphi_t(\boldsymbol{x}_0) \in K$ for any $t \geq 0$. Here $\varphi_t$ denotes the flow of the system.*

*Then, the system has a non-trivial closed orbit in $K$.*

Yes! The theorem seems to good to be true. In order to guarantee a periodic solution, one simply needs to exhibit a forward trajectory which is trapped inside $K$. This forward trajectory by itself needs not be periodic, but the theorem shows that if such a trajectory exists, then it will warrant a closed orbit for the system provided that $K$ fulfills the assumption of the theorem!

We will give the proof of the Poincaré-Bendixson's Theorem in the next section. Meanwhile let's go through some examples to illustrate the use of the theorem. One typical technique for applying the Poincaré-Bendixson's Theorem is to construct a **trapping region** in the phase portrait, so that trajectories starting from any point in the region will stay there for any positive time.

**Example 4.2.** Consider the system:

$$x' = x - y - x(x^2 + y^2)$$
$$y' = x + y - y(x^2 + y^2)$$

This is a system related to Hopf's Bifurcation (which we will not talk about in detail). However, due to this connection, let's call it the *Hopf's system* in this note.

While it seems difficult to solve the system using Cartesian coordinates, it is much nicer if one converts it into polar coordinates. Under the transformation rule $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$, we leave it as an exercise for readers to verify that the above system can be rewritten as:

$$r' = r(1 - r^2)$$
$$\theta' = 1$$

Therefore, if the initial data $\mathbf{x}_0$ is on the unit circle $r = 1$, then it will stay on it for all time. In polar coordinates, this solution can be explicitly written as $r(t) = 1$ and $\theta(t) = t + \theta_0$ where $\theta_0$ is the initial angle from the positive $x$-axis. Convert this solution back to Cartesian coordinates, it is written as: $\mathbf{x}(t) = (\cos(t + \theta_0), \sin(t + \theta_0))$. Clearly, $T = 2\pi$ is the period of the solution, i.e. $\mathbf{x}(t + 2\pi) = \mathbf{x}(t)$ for any $t \in \mathbb{R}$.

In general, if the initial data has polar coordinates $(r_0, \theta_0)$, then the solution to the system is given by

$$r(t) = \frac{e^t}{\sqrt{\left(\frac{1}{r_0^2} - 1\right) + e^{2t}}}, \quad \theta(t) = t + \theta_0.$$

Therefore, the trajectories off the unit circle are never periodic since $r(t)$ is either strictly decreasing (when $r_0 > 1$) or strictly increasing (when $0 < r_0 < 1$). In either case, the radius $r(t) \to 1$ as $t \to +\infty$. Therefore, these trajectories are approaching to the unit circle. See Figure 4 for the phase portrait.

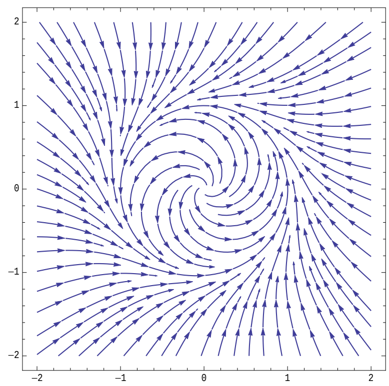


FIGURE 4. The phase portrait of the system in Example 4.2:
$$r' = r(1 - r^2), \quad \theta' = 1$$

The unit circle is the periodic solution to the system. However, let's **pretend we don't know this** and **try to use the Poincaré-Bendixson's Theorem to prove that a periodic solution exists**!

Let $K$ be the following closed and bounded subset of $\mathbb{R}^2$:

$$K = \left\{ \mathbf{x} \in \mathbb{R}^2 : \frac{1}{2} \leq |\mathbf{x}| \leq 2 \right\}$$

which is an annular region with outer radius 2 and inner radius $\frac{1}{2}$. The boundary of $K$ consists of a circle of radius $\frac{1}{2}$ and a circle of radius 2, both centered at the origin. See Figure 5.

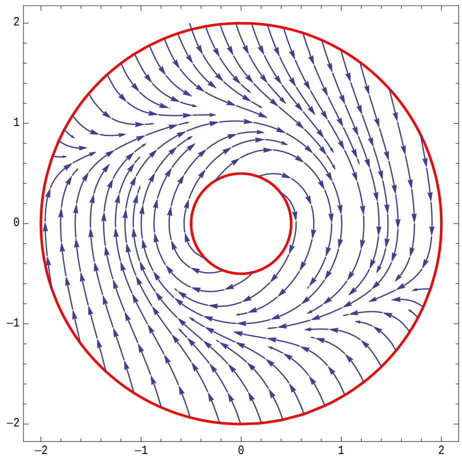

FIGURE 5. The solution curves starting from anywhere in $K$ are trapped inside $K$.

Under the system, $r' = r(1-r^2)$. Therefore, on the boundary circle $\{r = 2\}$, we have $r' = r(1-r^2) = -6 < 0$, and hence trajectories hitting $\{r = 2\}$ will decrease it's distance from the origin as $t$ increases. On the other hand, on another boundary circle $\{r = \frac{1}{2}\}$, we have $r' = r(1-r^2) = \frac{3}{8} > 0$, and hence trajectories hitting $\{r = \frac{1}{2}\}$ will increase $r$ as $t$ increases. These show any trajectories in the annular region $K$ will stay in $K$ for any future time.

Now that $K$ is closed and bounded. The only equilibrium point of the system, the origin, is not in $K$. From the above discussion, $K$ contains many forward trajectories (in particular it contains at least one). All these fulfill the conditions of the Poincaré-Bendixson's Theorem, so the system has a non-trivial closed orbit in $K$. Of course, the closed orbit as we figured out before is the unit circle.

$\square$

**Example 4.3.** Back to the glycolysis system:

$$x' = -x + ay + x^2 y$$
$$y' = b - ay - x^2 y$$

where $a,\ b > 0$ are two parameters.

We will show the quadrilateral region $K$ with vertex $(0,0)$, $(b + \frac{b}{a}, 0)$, $(b, \frac{b}{a})$ and $(0, \frac{b}{a})$ is a trapping region for the system. See Figure 6 for the sketch of the region.

To show it is a trapping region, we need to show the vector field $\mathbf{F}(x,y) = \begin{bmatrix} -x + ay + x^2 y \\ b - ay - x^2 y \end{bmatrix}$ is pointing into the region near the boundary, or equivalently, $\mathbf{F} \cdot \mathbf{n} > 0$ where $\mathbf{n}$ is an inward normal vector of the boundary.

There are four boundary components, three of which are either horizontal or vertical. Let's verify two of them and the other two are left as an exercise.

On the boundary segment joining $(0,0)$ and $(b + \frac{b}{a}, 0)$, we have $y = 0$ (when $x$ is varying) and the inward normal vector $\mathbf{n}$ is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and we have:

$$\mathbf{F}(x,0) \cdot \mathbf{n} = (-x, b) \cdot (0,1) = b > 0.$$

Hence $\mathbf{F}$ is pointing inward on this boundary component.

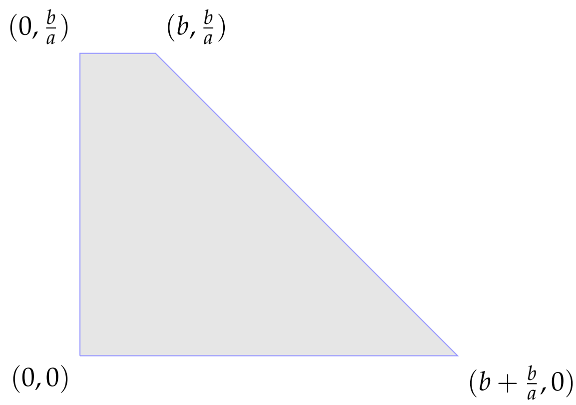


FIGURE 6. The trapping region in Example 4.3.

The boundary component joining $(b+\frac{b}{a}, 0)$ and $(b, \frac{b}{a})$ can be expressed as $y = -x+b+\frac{b}{a}$, with $x \in [b, b+\frac{b}{a}]$, and the inward normal vector $\mathbf{n}$ is $(-1,-1)$. Therefore,

$$\mathbf{F}(x,y)\cdot\mathbf{n} = -(-x+ay+x^2y) - (b-ay-x^2y) = x-b \geq 0$$

since $x \geq b$.

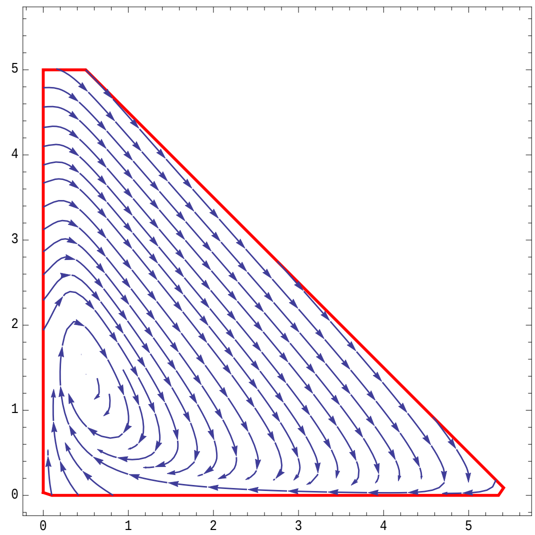


FIGURE 7. The phase portrait inside the trapping region in Example 4.3 when $(a,b) = (\frac{1}{10}, \frac{1}{2})$.

After verifying the other two boundary components, we can conclude $K$ is a trapping region. See Figure 7 for the phase portrait inside the trapping region for this pair of $(a,b)$. Clearly $K$ is closed and bounded. Unfortunately, there is an equilibrium point $(b, \frac{b}{a+b^2})$ which is inside $K$! One cannot apply the Poincaré-Bendixson's Theorem with this $K$ directly. However, it is still possible to show existence of periodic solution if $(b, \frac{b}{a+b^2})$ can be shown to be unstable, since then one can drill a small open ball $B_\varepsilon((b, \frac{b}{a+b^2}))$ inside $K$ and $K \backslash B_\varepsilon((b, \frac{b}{a+b^2}))$ is a closed and bounded trapping region for the system not containing any equilibrium point. The Poincaré-Bendixson's Theorem hence shows there is a periodic solution inside the region $K \backslash B_\varepsilon((b, \frac{b}{a+b^2}))$.

The equilibrium point can be shown to be unstable for some (although not all) pairs of $(a, b)$, for instance $a = \frac{1}{10}$ and $b = \frac{1}{2}$ (one can use linearization to show that). There are many other such pairs of $a$ and $b$ too. $\square$

## 5. Poincaré-Bendixson's Theorem: the proof

This section is devoted to the proof of the Poincaré-Bendixson's Theorem. It is an amazingly beautiful and intelligent proof, using a celebrated result in topology called the Jordan Curve Theorem.

5.1. **Limit Sets.** An important concept in the proof of the Poincaré-Bendixson's Theorem is the $\alpha-$ and $\omega-$ limit sets to be defined below.

Let $\varphi_t$ be the flow of the Hopf's system discussed in Example 4.2. Consider the trajectory $\varphi_t(\mathbf{x}_0)$ for some point $\mathbf{x}_0 \in \mathbb{R}^2$ with polar coordinates $(r_0, 0)$. As we computed before, the trajectory is given in polar coordinates by:

$$r(t) = \frac{e^t}{\sqrt{\left(\frac{1}{r_0^2} - 1\right) + e^{2t}}}, \quad \theta(t) = t,$$

or equivalently in $(x, y)$-coordinates:

$$\varphi_t(\mathbf{x}_0) = (x(t), y(t)) = \frac{e^t}{\sqrt{\left(\frac{1}{r_0^2} - 1\right) + e^{2t}}}(\cos t, \sin t).$$

Although the scaling factor $\frac{e^t}{\sqrt{\left(\frac{1}{r_0^2} - 1\right) + e^{2t}}}$ approaches to 1 as $t \to +\infty$, the limit $\lim_{t\to+\infty} \varphi_t(\mathbf{x}_0)$ does not exist because the trigonometric functions $\cos t$ and $\sin t$ are oscillating between $-1$ and 1 rather than converging to specific numbers.

However, if one substitute $t$ by a suitable time sequence $\{t_n\}_{n=1}^{\infty}$ which approaches to $+\infty$ as $n \to \infty$, then one can possibly talk about convergence of $\varphi_{t_n}(\mathbf{x}_0)$ as $n \to \infty$. For example, if we let $t_n = 2\pi n$, then

$$\begin{aligned}
\varphi_{t_n}(\mathbf{x}_0) &= \frac{e^{2\pi n}}{\sqrt{\left(\frac{1}{r_0^2} - 1\right) + e^{4\pi n}}}(\cos(2\pi n), \sin(2\pi n)) \\
&= \frac{e^{2\pi n}}{\sqrt{\left(\frac{1}{r_0^2} - 1\right) + e^{4\pi n}}}(1, 0).
\end{aligned}$$

Letting $n \to \infty$ gives $\varphi_{t_n}(\mathbf{x}_0) \to (1, 0)$ as $n \to \infty$.

That says, although we do not have convergence for $\varphi_t(\mathbf{x}_0)$ when $t$ is regarded as a *continuous* parameter, we can still talk about a *discrete* notion of convergence by substituting $t$ by a suitable sequence $t_n$. The cost is that now the "limit" may not be unique. For instance, if one choose $t_n = 2\pi n + \theta_0$ where $\theta_0$ is any fixed angle, then one should verify that $\varphi_{t_n}(\mathbf{x}_0) \to (\cos\theta_0, \sin\theta_0)$ as $n \to +\infty$, which is another point on the unit circle.

Under this generalized notion of limits, we no longer say $\varphi_t(\mathbf{x}_0)$ converges to a particular *point*, but rather say $\varphi_t(\mathbf{x}_0)$ approaches to a *set*. This motivates the following definition:

**Definition 5.1** (Limit Points and Limit Sets)**.** Let $\varphi_t$ be the flow of an ODE system on $\mathbb{R}^d$. A point $\mathbf{y} \in \mathbb{R}^d$ is called an **$\omega$-limit point** of $\mathbf{x}_0$ if there exists a time sequence $t_n \to +\infty$ as $n \to \infty$ such that $\varphi_{t_n}(\mathbf{x}_0) \to \mathbf{y}$ as $n \to \infty$. The **$\omega$-limit set of** $\mathbf{x}_0$, denoted by $\omega(\mathbf{x}_0)$, is the set of all possible $\omega$-limit points of $\mathbf{x}_0$. Precisely,

$$\omega(\mathbf{x}_0) = \{\mathbf{y} \in \mathbb{R}^d : \exists t_n \to +\infty \text{ as } n \to \infty \text{ such that } \varphi_{t_n}(\mathbf{x}_0) \to \mathbf{y} \text{ as } n \to \infty\}.$$

A point $\mathbf{z} \in \mathbb{R}^d$ is called an **$\alpha$-limit point** of $\mathbf{x}_0$ if there exists a time sequence $t_n \to -\infty$ as $n \to \infty$ such that $\varphi_{t_n}(\mathbf{x}_0) \to \mathbf{y}$ as $n \to \infty$. The **$\alpha$-limit set of** $\mathbf{x}_0$, denoted by $\alpha(\mathbf{x}_0)$, is the set of all possible $\alpha$-limit points of $\mathbf{x}_0$. Precisely,

$$\alpha(\mathbf{x}_0) = \{\mathbf{z} \in \mathbb{R}^d : \exists t_n \to -\infty \text{ as } n \to \infty \text{ such that } \varphi_{t_n}(\mathbf{x}_0) \to \mathbf{z} \text{ as } n \to \infty\}.$$

□

*Remark* 5.2. The letters $\alpha$ and $\omega$ are chosen because they are the first and the last Greek alphabet respectively. □

**Example 5.3.** Recall that the flow of the Hopf's system is given by

$$\varphi_t(\mathbf{x}_0) = (x(t), y(t)) = \frac{e^t}{\sqrt{\left(\frac{1}{r_0^2} - 1\right) + e^{2t}}}(\cos t, \sin t),$$

where $\mathbf{x}_0 = (r_0, 0)$ in $(x, y)$-coordinates and $r_0 > 0$.

As discussed before, there exists a sequence of times $t_n = 2\pi n + \theta_0 \to +\infty$ as $n \to \infty$ such that $\varphi_{t_n}(\mathbf{x}_0) \to (\cos\theta_0, \sin\theta_0)$. One can pick $\theta_0$ to be any angle, so any point on the unit circle is an $\omega$-limit point of $\mathbf{x}_0$. Conversely, any $\omega$-limit point of $\mathbf{x}_0$ must be on the unit circle since $|\varphi_{t_n}(\mathbf{x}_0)| \to 1$ as $n \to \infty$ for any sequence $t_n \to +\infty$. Therefore, the $\omega$-limit set of $\mathbf{x}_0$ is the unit circle. Symbolically, we denote it by:

$$\omega(\mathbf{x}_0) = \{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{y}| = 1\}.$$

The $\alpha$-limit points of $\mathbf{x}_0$ is bit more subtle than their $\omega$-counterparts. If $\mathbf{x}_0 = (r_0, 0)$ is chosen such that $0 < r_0 < 1$, then $\frac{1}{r_0^2} - 1$ is positive and so $\sqrt{\left(\frac{1}{r_0^2} - 1\right) + e^{2t}}$ is defined for all time $t$. Therefore it makes sense to talk about $\varphi_{t_n}(\mathbf{x}_0)$ for sequences $t_n \to -\infty$. One can verify that in this case $\varphi_{t_n}(\mathbf{x}_0) \to (0, 0)$ for any sequence $t_n \to -\infty$, and so $\mathbf{0}$ is the only $\alpha$-limit point of $\mathbf{x}_0 = (r_0, 0)$. Symbolically, it is defined by:

$$\alpha(\mathbf{x}_0) = \{\mathbf{0}\} \quad \text{when } 0 < r_0 < 1.$$

However, $\mathbf{x}_0 = (r_0, 0)$ with $r_0 > 1$. The square-root $\sqrt{\left(\frac{1}{r_0^2} - 1\right) + e^{2t}}$ is undefined when $t$ is sufficiently negative. It is therefore forbidden to substitute $t$ by a sequence $t_n$ that goes to $-\infty$. Therefore, there is no $\alpha$-limit point for this $\mathbf{x}_0$, and symbolically we say:

$$\alpha(\mathbf{x}_0) = \emptyset \quad \text{when } r_0 > 1.$$

□

We will mostly deal with $\omega$-limits in the rest of the chapter. While it is possible to determine the limit sets for the Hopf's system where the flow map can be explicitly stated, it is in general difficult to determine limit sets for most nonlinear systems. In the rest of the chapter, we will deal with limit sets in a qualitative way rather than finding them explicitly.

The following lemma presents some important facts about $\omega$-limit sets. They will be used often when establishing the Poincaré-Bendixson's Theorem.

**Lemma 5.4.** *Let $\varphi_t$ be the flow of a $C^1$-system on $\mathbb{R}^d$. Suppose $\boldsymbol{y} \in \omega(\boldsymbol{x})$ for some $\boldsymbol{x} \in \mathbb{R}^d$, then we have:*

*(1) $\varphi_s(\boldsymbol{y}) \in \omega(\boldsymbol{x})$ for any $s$ (as long as $\varphi_s(\boldsymbol{y})$ exists).*
*(2) If $\boldsymbol{z} = \varphi_s(\boldsymbol{y})$ for some fixed $s$, i.e. $\boldsymbol{z}$ is on the trajectory through $\boldsymbol{y}$, then $\boldsymbol{z} \in \omega(\boldsymbol{x})$.*
*(3) If $\boldsymbol{w} \in \omega(\boldsymbol{y})$, i.e. $\boldsymbol{w}$ is an $\omega$-limit point of $\boldsymbol{y}$, then we also have $\boldsymbol{w} \in \omega(\boldsymbol{x})$. [In other words, $\boldsymbol{w} \in \omega(\boldsymbol{y})$ and $\boldsymbol{y} \in \omega(\boldsymbol{x})$ imply $\boldsymbol{w} \in \omega(\boldsymbol{x})$.]*

*Proof.* Given that $\mathbf{y} \in \omega(\mathbf{x})$, there exists a sequence of times $t_n \to +\infty$ such that $\lim_{n\to\infty} \varphi_{t_n}(\mathbf{x}) = \mathbf{y}$.

Parts (1) and (2) are easy consequences of the continuity of $\varphi_s$. The detail is as follows:

To prove (1), we consider $\varphi_{s+t_n}(\mathbf{y}) = \varphi_s(\varphi_{t_n}(\mathbf{y}))$. Since $\varphi_s$ is continuous, we have:

$$\lim_{n\to\infty} \varphi_{s+t_n}(\mathbf{x}) = \lim_{n\to\infty} \varphi_s(\varphi_{t_n}(\mathbf{x})) = \varphi_s\left(\lim_{n\to\infty} \varphi_{t_n}(\mathbf{x})\right) = \varphi_s(\mathbf{y}).$$

Therefore, $\varphi_s(\mathbf{y}) \in \omega(\mathbf{x})$ and the associated time sequence is $s + t_n$.

For (2), we consider:

$$\begin{aligned}
z &= \varphi_s(\mathbf{y}) && \text{(given)}\\
&= \varphi_s\left(\lim_{n\to\infty} \varphi_{t_n}(\mathbf{x})\right) && \text{(given)}\\
&= \lim_{n\to\infty} \varphi_s(\varphi_{t_n}(\mathbf{x})) && (\varphi_s \text{ is continuous})\\
&= \lim_{n\to\infty} \varphi_{t_n+s}(\mathbf{x}).
\end{aligned}$$

Therefore, $z$ is an $\omega$-limit point of $\mathbf{x}$ since there exists a time sequence $t_n + s \to +\infty$ such that $\varphi_{t_n+s}(\mathbf{x}) \to \mathbf{z}$. It completes the proof of (2).

For (3), since $\mathbf{w} \in \omega(\mathbf{y})$ there exists a sequence of times $s_k \to \infty$ such that $\varphi_{s_k}(\mathbf{y}) \to \mathbf{w}$ as $k \to \infty$. Given that $\lim_{n\to\infty} \varphi_{t_n}(\mathbf{x}) = \mathbf{y}$ and by the continuity of $\varphi_{s_k}$, we have $\lim_{n\to\infty} \varphi_{s_k+t_n}(\mathbf{x}) = \varphi_{s_k}(\mathbf{y})$ for each fixed $k$. We pick a subsequence $t_{n_k}$ of $t_n$ such that for each $k$, we have

$$\left|\varphi_{s_k+t_{n_k}}(\mathbf{x}) - \varphi_{s_k}(\mathbf{y})\right| < \frac{1}{k}.$$

Consider the sequence of times $s_k + t_{n_k}$, we then have:

$$\begin{aligned}
\left|\varphi_{s_k+t_{n_k}}(\mathbf{x}) - \mathbf{w}\right| &= \left|\varphi_{s_k+t_{n_k}}(\mathbf{x}) - \varphi_{s_k}(\mathbf{y}) + \varphi_{s_k}(\mathbf{y}) - \mathbf{w}\right|\\
&\leq \left|\varphi_{s_k+t_{n_k}}(\mathbf{x}) - \varphi_{s_k}(\mathbf{y})\right| + \left|\varphi_{s_k}(\mathbf{y}) - \mathbf{w}\right|\\
\left|\varphi_{s_k+t_{n_k}}(\mathbf{x}) - \mathbf{w}\right| &\leq \underbrace{\frac{1}{k}}_{\to 0} + \underbrace{\left|\varphi_{s_k}(\mathbf{y}) - \mathbf{w}\right|}_{\to 0}.
\end{aligned}$$

As $k \to +\infty$, we have $\left|\varphi_{s_k+t_{n_k}}(\mathbf{x}) - \mathbf{w}\right| \to 0$, or in other words $\varphi_{s_k+t_{n_k}}(\mathbf{x}) \to \mathbf{w}$. Therefore, $\mathbf{w} \in \omega(\mathbf{x})$ and its associated time sequence is $s_k + t_{n_k}$. □

5.1.1. *Closedness and boundedness of the trapping region.* Recall there are two conditions for the trapping region $K$ in the statement of the Poincaré-Bendixson's Theorem, namely $K$ has to be closed and bounded. These two conditions have important implications in terms of limit sets.

Suppose $\varphi_t(\mathbf{x}_0)$ is a forward trajectory contained in $K$ entirely. If $K$ were not bounded, then $\varphi_t(\mathbf{x}_0)$ may diverge to infinity as $t \to +\infty$ then there is no $\omega$-limit point to talk about. The boundedness of $K$ guarantees there is at least one $\omega$-limit point of $\mathbf{x}_0$. In fact, it is a consequence of the following famous theorem in analysis:

**Theorem 5.5** (Bolzano-Weierstrass's Theorem)**.** *If $S$ is a bounded infinite set in $\mathbb{R}^d$, then there exists a sequence $\mathbf{s}_n \in S$ such that $\mathbf{s}_n$ converges to a limit $\mathbf{s}_0$ in $\mathbb{R}^d$ as $n \to \infty$.*

We omit the proof here. A standard proof can be found in any basic analysis textbook and should have been covered in MATH 2031.

Now applying the Bolzano-Weierstrass's Theorem to our scenario. The forward trajectory $\varphi_t(\mathbf{x}_0)$ is an infinite set (unless $\mathbf{x}_0$ is an equilibrium point, but then $\omega(\mathbf{x}_0)$ is $\{\mathbf{x}_0\}$ itself). If it is completely inside a bounded set $K$, then the trajectory is a bounded infinite set so the theorem implies there exists a sequence $\varphi_{t_n}(\mathbf{x}_0)$ that converges to a limit $\mathbf{y}$ in $\mathbb{R}^d$. Consequently, $\omega(\mathbf{x})$ contains at least one point $\mathbf{y}$.

The boundedness of $K$ guarantees the forward trajectory has at least one $\omega$-limit point. However, boundedness *alone* cannot guarantee the limit point must be in $K$. That's why we need to combine closedness with boundedness. The following is a "common-sense" fact in analysis and point-set topology:

**Proposition 5.6.** *Let $K$ be a closed set in $\mathbb{R}^d$. If $\mathbf{x}_n$ is a sequence in $K$ and that $\mathbf{x}_n \to \mathbf{y}$ as $n \to \infty$, then the limit $\mathbf{y}$ must be in $K$.*

*Proof.* We prove by contradiction. Suppose $\mathbf{y}$ is not in $K$, then $\mathbf{y} \in \mathbb{R}^d \backslash K$. Since $K$ is closed, the complement $\mathbb{R}^d \backslash K$ is open. By the definition of openness, there exists a ball $B_\varepsilon(\mathbf{y})$ that is contained inside $\mathbb{R}^d \backslash K$.

The sequence $\mathbf{x}_n \to \mathbf{y}$ as $n \to \infty$, so $\mathbf{x}_n$ will eventually enter the ball $B_\varepsilon(\mathbf{y})$ for sufficiently large $n$. However, it is not possible since all $\mathbf{x}_n$'s are in $K$ but the ball $B_\varepsilon(\mathbf{y})$ is disjoint from $K$. Therefore, we must have $\mathbf{y} \in K$. □

Combining closedness and boundedness of $K$, the forward trajectory $\varphi_t(\mathbf{x}_0)$ trapped inside $K$ must have at least one $\omega$-limit point, and all $\omega$-limit points must be in $K$. This is significant since then our proof "game" will be confined in the trapping region $K$.

5.2. **Local Sections and Flow Boxes.** The Poincaré-Bendixson's Theorem requires the trapping region $K$ has no equilibrium point for the system. We will explore why this is needed in this subsection.

From now on, we will restrict the discussion to planar system only. Let $\mathbf{x}_0$ be a non-equilibrium point of a $C^1$-system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$. The vector $\mathbf{F}(\mathbf{x}_0)$ at $\mathbf{x}_0$ is non-zero, and so there is a straight line, denoted by $l(\mathbf{x}_0)$, passing through $\mathbf{x}_0$ and is perpendicular to $\mathbf{F}(\mathbf{x}_0)$. Pick a point $\mathbf{x}$ on this line $l(\mathbf{x}_0)$, then one can tell whether $\mathbf{F}(\mathbf{x})$ is pointing at the same side of the line as $\mathbf{F}(\mathbf{x}_0)$ by considering the dot product $\mathbf{F}(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}_0)$. If the dot product is positive, then $\mathbf{F}(\mathbf{x})$ points at the same side of the line as $\mathbf{F}(\mathbf{x}_0)$.

Since $\mathbf{F}(\mathbf{x}_0) \cdot \mathbf{F}(\mathbf{x}_0) = |\mathbf{F}(\mathbf{x}_0)|^2 > 0$, by continuity of the vector field, the dot product $\mathbf{F}(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}_0)$ must be positive as far as $\mathbf{x}$ is sufficiently close to $\mathbf{x}_0$. Consequently, one can find a line segment $\mathcal{S}_{\mathbf{x}_0}$ of $l(\mathbf{x}_0)$ such that at every point $\mathbf{x}$ on this line segment $\mathcal{S}_{\mathbf{x}_0}$, the vector field $\mathbf{F}(\mathbf{x})$ is pointing at the same side of $l(\mathbf{x}_0)$ as $\mathbf{F}(\mathbf{x}_0)$. This line segment is called:

**Definition 5.7** (Local Sections)**.** Let $\mathbf{x}_0$ be a non-equilibrium point of a $C^1$-system $\mathbf{x}' = \mathbf{F}(\mathbf{x})$. A **local sections** $\mathcal{S}_{\mathbf{x}_0}$ is a line segment passing through $\mathbf{x}_0$ and perpendicular to $\mathbf{F}(\mathbf{x}_0)$ such that $\mathbf{F}(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}_0) > 0$ for any $\mathbf{x} \in \mathcal{S}_{\mathbf{x}_0}$. □

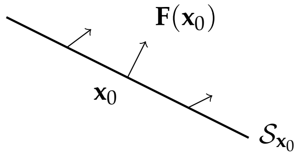


FIGURE 8. A local section $\mathcal{S}_{\mathbf{x}_0}$ based at $\mathbf{x}_0$.

Given a local section $\mathcal{S}_{\mathbf{x}_0}$, one can construct a **flow box** at $\mathbf{x}_0$ to be described below. As the vector field $\mathbf{F}(\mathbf{x})$ is pointing at the same side of the local section $\mathcal{S}_{\mathbf{x}_0}$ when $\mathbf{x}$ is sufficiently close to $\mathbf{x}_0$. One can expect there is a neighborhood $\mathcal{V}$ of $\mathbf{x}_0$ so that the trajectories inside $\mathcal{V}$ are *flowing in approximately parallel directions* as shown in Figure 9. A flow box has two edges: the in-edge and the out-edge. A flow box is characterized by the following properties:

(1) Any trajectory must enter the flow box $\mathcal{V}$ through its in-edge.
(2) After a trajectory enters the flow box $\mathcal{V}$, it must intersect the local section $\mathcal{S}_{\mathbf{x}_0}$ exactly once before leaving $\mathcal{V}$.
(3) Any trajectory must leave the flow box $\mathcal{V}$ through its out-edge.

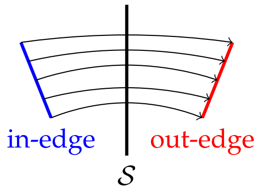


FIGURE 9. An example of a flow box and a local section.

The formal construction of flow boxes is bit technical so we omit it here. Readers may consult Section 10.2 of Hirsch-Smale-Devaney's book for both the formal definition and the existence proof of flow boxes using the Implicit Function Theorem. In order to understand the key idea of the Poincaré-Bendixson's Theorem, it is more important to keep in mind the *geometric intuition* of flow boxes, rather than knowing the formal definition or why flow boxes must exist.

5.3. **Jordan Curve Theorem and Consequences.** Limit sets, local sections and flow boxes are three key ingredients in the proof of the Poincaré-Bendixson's Theorem. In this subsection, we will first state (but not prove) a celebrated result in topology, the Jordan Curve Theorem. It will lead to two important consequences about limit sets and local sections.

The statement of the Jordan Curve Theorem, stated below, sounds quite trivial and you may wonder why such an obvious statement can be qualified as a theorem. Nonetheless, the proof requires an advanced concept called Homology which is usually taught in graduate level Algebraic Topology course.

**Theorem 5.8** (Jordan Curve Theorem)**.** *Any continuous simple closed curve $C$ in the plane $\mathbb{R}^2$ divides the plane into two disjoint components, i.e. there exist two disjoint connected open sets $U$ and $V$ such that $\mathbb{R}^2\backslash C = U \cup V$. Moreover, one of the $U$ and $V$ is bounded and the other one must be unbounded.*

The first consequence of the Jordan Curve Theorem is about monotonicity:

**Lemma 5.9.** *Let $\varphi_t$ be the flow of a $C^1$ planar system, and let $\mathcal{S}$ be any local section. Consider a trajectory $\varphi_t(\boldsymbol{y})$ from a point $\boldsymbol{y}$ in $\mathbb{R}^2$. If $t_1 < t_2 < t_3$ are times at which the trajectory $\varphi_t(\boldsymbol{y})$ intersects $\mathcal{S}$, then the intersection points $\varphi_{t_1}(\boldsymbol{y})$, $\varphi_{t_2}(\boldsymbol{y})$ and $\varphi_{t_3}(\boldsymbol{y})$ must be in monotonic order on the local section $\mathcal{S}$ (see Figures 10 and 11 for an example and a non-example of monotonically ordered points).*

*Proof.* First construct a continuous simple closed curve $C$ by gluing the part of the trajectory $\varphi_t(\mathbf{y})$ for $t \in [t_1, t_2]$ and the line segment joining $\varphi_{t_1}(\mathbf{y})$ and $\varphi_{t_2}(\mathbf{y})$. The Jordan Curve Theorem asserts that $C$ divides the plane $\mathbb{R}^2$ into two disjoint open sets $U$ and $V$. Assume without loss of generality that the trajectory $\varphi_t(\mathbf{y})$ enters the region $U$ shortly after $t_2$. Suppose at a later time $t_3$, the trajectory intersects $\mathcal{S}$ in the middle of $\varphi_{t_1}(\mathbf{y})$ and $\varphi_{t_2}(\mathbf{y})$ (let's call this $1-3-2$ configuration), then $\mathcal{S}$ being a local section implies the trajectory must come from another region $V$ shortly before $t_3$ (see Figure 12). However, it is impossible since $U$ and $V$ are disjoint. It rules out the $1-3-2$ arrangement on the local section $\mathcal{S}$. Similarly, one can also rule out the $3-1-2$ configuration by the same argument. Therefore, the only possibility is $1-2-3$, which is exactly what we need to show. $\square$

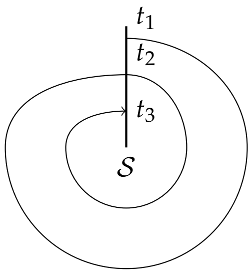


FIGURE 10. $\varphi_{t_i}(\mathbf{x}_0)$'s are monotonically ordered on $\mathcal{S}$

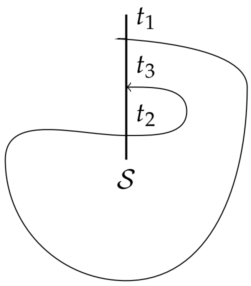


FIGURE 11. $\varphi_{t_i}(\mathbf{x}_0)$'s are **not** monotonically ordered on $\mathcal{S}$

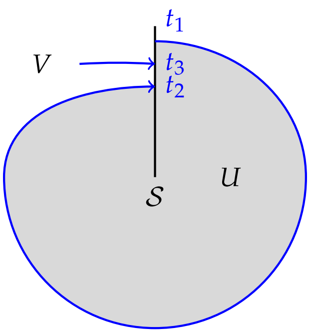


FIGURE 12. The trajectory in blue is a hypothetical trajectory that gives a $1-3-2$ configuration. This configuration is ruled out by the Jordan Curve Theorem.

If we further assume that the point $\mathbf{y}$ of Lemma 5.9 is an $\omega$-limit point of another point $\mathbf{x}$, then we have a stronger result:

**Lemma 5.10.** *Let $\varphi_t$ be the flow of a $C^1$ planar system, and let $\mathcal{S}$ be any local section. If a trajectory $\varphi_t(\boldsymbol{y})$ starts from a point $\boldsymbol{y} \in \omega(\boldsymbol{x})$ for some $\boldsymbol{x} \in \mathbb{R}^2$, then the trajectory $\varphi_t(\boldsymbol{y})$ intersects $\mathcal{S}$ at at most one point.*

*Remark* 5.11. The trajectory can intersect $\mathcal{S}$ for infinitely many *times*, but the lemma shows the intersection *point* must be the same every time. □

*Proof.* We prove by contradiction. Suppose $\varphi_t(\mathbf{y})$ intersects $\mathcal{S}$ at two different points $\mathbf{z}$ and $\mathbf{w}$. One can then find two disjoint flow boxes, $\mathcal{V}$ based at $\mathbf{z}$ and another $\mathcal{W}$ based at $\mathbf{w}$.

As $\mathbf{z}$ and $\mathbf{w}$ are on the trajectory from $\mathbf{y}$, they are both $\omega$-limit points of $\mathbf{x}$ by Lemma 5.4. As a result, there exist sequences $t_n \to +\infty$ and $s_n \to +\infty$ such that $\varphi_{t_n}(\mathbf{x}) \to \mathbf{z}$ and $\varphi_{s_n}(\mathbf{x}) \to \mathbf{w}$ as $n \to \infty$.

There must be infinitely many $\varphi_{t_n}(\mathbf{x})$'s in $\mathcal{V}$ since $\varphi_{t_n}(\mathbf{x})$ converges to $\mathbf{z}$ which is inside $\mathcal{V}$. Therefore, by the property of a flow box, the trajectory $\varphi_t(\mathbf{x})$ must enter the flow box for infinitely many times and intersect the $\mathcal{V}$-portion of the local section $\mathcal{S}$ for infinitely many times. See Figure 13.

Similarly, the trajectory $\varphi_t(\mathbf{x})$ must intersect the $\mathcal{W}$-portion of the local section $\mathcal{S}$ for infinitely many times. However, Lemma 5.9 shows $\varphi_t(\mathbf{x})$ must intersect the local section $\mathcal{S}$ in monotonic order. It is impossible to have this trajectory intersecting the $\mathcal{V}$- and $\mathcal{W}$-portions of the local section $\mathcal{S}$ both for infinitely many times and overall in a monotonic order. It leads to a contradiction. Therefore, $\varphi_t(\mathbf{y})$ cannot intersect $\mathcal{S}$ at two different points, and hence it can only intersect $\mathcal{S}$ at at most one point. □

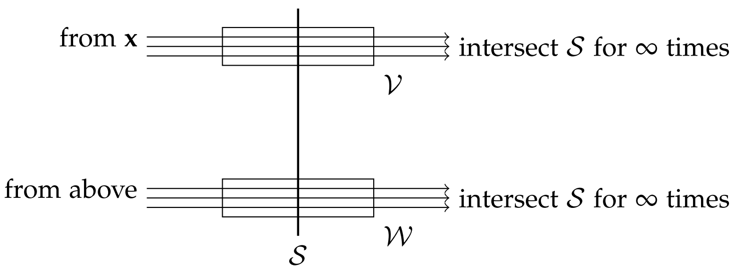


FIGURE 13. The trajectory from $\mathbf{x}$ cannot intersect $\mathcal{S}$ first in $\mathcal{V}$ and then $\mathcal{W}$ both for infinitely many times in a monotonic manner. This leads to a contradiction. Note that the trajectory from $\mathbf{y}$ is not shown in the figure since it is not relevant.

5.4. **Completion of the Proof: A Tale of Three Points.** Finally, with Lemma 5.10, we are ready to give the proof of the Poincaré-Bendixson's Theorem.

*Proof of Poincaré-Bendixson's Theorem.* Recall that the set-up is that there is a closed and bounded set $K$ in $\mathbb{R}^2$ that contains a forward trajectory $\varphi_t(\mathbf{x})$ for $t \in [0, \infty)$. By the closedness and boundedness of $K$, the limit set $\omega(\mathbf{x})$ is non-empty (by Bolzano-Weierstrass) and is contained inside $K$ (by closedness). The absence of equilibrium point in $K$ and that $\omega(\mathbf{x}) \subset K$ guarantee every point on $\omega(\mathbf{x})$ has a local section and a flow box around the point.

Now let $\mathbf{y}$ be any point in $\omega(\mathbf{x})$. **The key idea of proving the theorem is to show that $\varphi_t(\mathbf{y})$ is a periodic solution**. In order to prove this, consider a point $\mathbf{z} \in \omega(\mathbf{y})$.

Since $\mathbf{y} \in \omega(\mathbf{x})$, Lemma 5.4 shows $\varphi_t(\mathbf{y}) \in \omega(\mathbf{x}) \subset K$ for any $t \geq 0$. Consequently, the closedness of $K$ implies $\omega(\mathbf{y}) \subset K$, and so $\mathbf{z} \in K$.

Now that $\mathbf{z} \in K$, it is not an equilibrium point and so there exists a local section $\mathcal{S}$ and a flow box $\mathcal{V}$ based at $\mathbf{z}$. Now the proof is completed by applying Lemma 5.10: since $\mathbf{y}$ is an $\omega$-limit point of $\mathbf{x}$, the lemma shows that the trajectory $\varphi_t(\mathbf{y})$ intersects the local section $\mathcal{S}$ at at most one point. Since $\mathbf{z}$ is an $\omega$-limit point of $\mathbf{y}$, it implies $\varphi_t(\mathbf{y})$ must enter the flow box $\mathcal{V}$ for at infinitely many times, and intersect $\mathcal{S}$ for infinitely many times but every time the intersection point must be the same. Therefore, one can pick two different times $s$ and $t$, where $s < t$, such that $\varphi_t(\mathbf{y}) = \varphi_s(\mathbf{y})$, which implies $\varphi_{t-s}(\mathbf{y}) = \mathbf{y}$. In other words, the trajectory $\varphi_t(\mathbf{y})$ is periodic with a period $t - s > 0$. It completes the proof. □

## References

[1] Hirsch, Morris W., Smale, Stephen, and Devaney, Robert L., *LaTeX: Differential equations, dynamical systems, and an introduction to chaos*, Third edition. Elsevier/Academic Press, Amsterdam, 2013. xiv+418 pp. ISBN: 978-0-12-382010-5